Eigenvalue bounds for the independence and chromatic number of graph powers and its applications

Aida Abiad
(TU/e, UGent, VUB)

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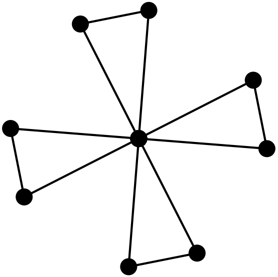
Rijeka Conference on Combinatorial Objects and their Applications

Plan

Adjacency eigenvalues



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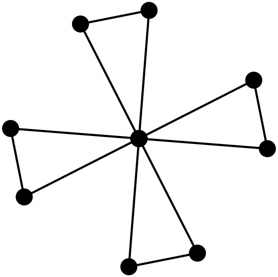
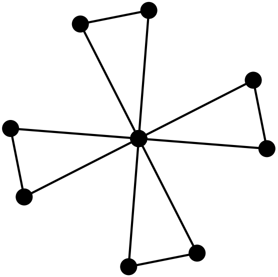


Dutch windmill graph

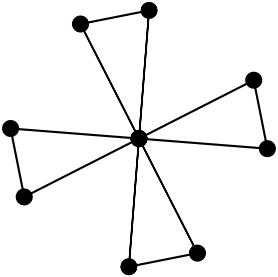
Adjacency eigenvalues



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & \cdots \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & \\ \vdots & & & & & & & \end{bmatrix} \end{array}$$

Dutch windmill graph (or friendship graph)

Adjacency eigenvalues



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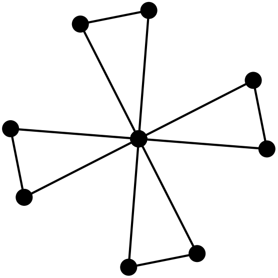
$$\lambda_1 \geq \cdots \geq \lambda_n$$

Adjacency eigenvalues



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$$\{\theta_0, \ldots, \theta_d\}$$

Adjacency matrix and closed walks

Adjacency matrix $A = (a_{ij})$

Power adjacency matrix $A^k = (a_{ij}^k)$

$a_{ij}^k = \#$ walks of length k from i to j

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algebraic

combinatorics

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  (Fiol and Garriga 1998) If G is k-partially walk-regular, for any polynomial $p \in \mathbb{R}_k[x]$, the diagonal of $p(A)$ is constant with entries

$$(p(A))_{uu} = \frac{1}{n} \operatorname{tr} p(A) = \frac{1}{n} \sum_{i=1}^{n} p(\lambda_i) \quad \text{for all } u \in V.$$

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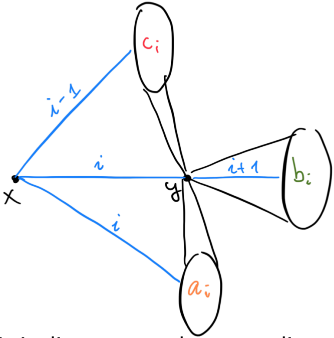
  Every graph is k-partially walk-regular for $k = 0, 1$, and every regular graph is 2-partially walk-regular.

- G is k-partially walk-regular for any k iff G is walk-regular.

k-partially distance-regular

- G of diameter D is **distance-regular** if there are constants c_i, a_i, b_i such that for all $i = 0, 1, \ldots, D$, and all vertices x and y at distance $i = d(x, y)$, among the neighbors of y, there are
  - c_i at distance $i - 1$ from x,
  - a_i at distance i,
  - b_i at distance $i + 1$.



- G is k-**partially distance-regular** if it is distance-regular up to distance k.

Graph power G^k

The k^{th} **power of a graph** $G = (V, E)$, denoted by G^k, is formed by connecting two vertices if they are at distance at most k.

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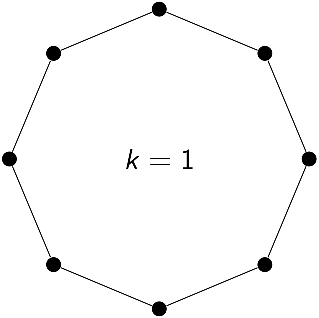

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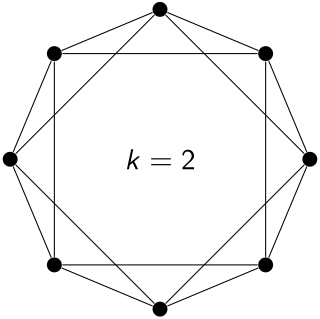

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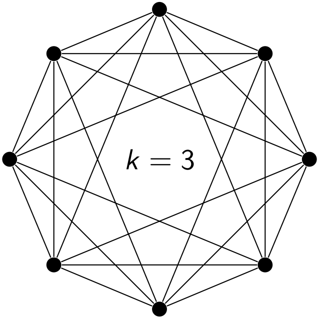

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Quite some work on powers of graphs, e.g. (Alon and Mohar 2002) and (Atkinson and Frieze 2004).

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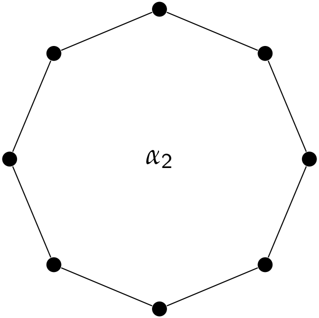

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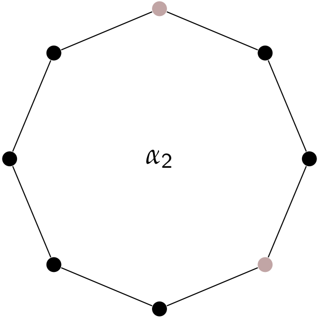

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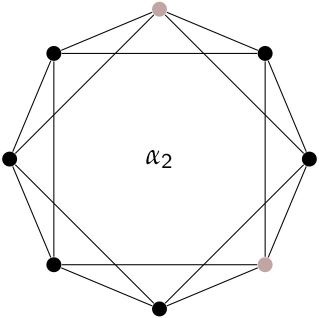

Note: $\alpha_k(G) = \alpha(G^k)$

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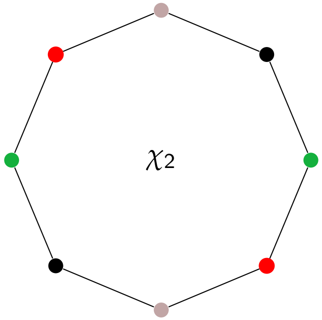
(Kramer and Kramer 1969) χ_k: smallest number of colours required to colour all the vertices of G, such that no two vertices within distance k share a colour.



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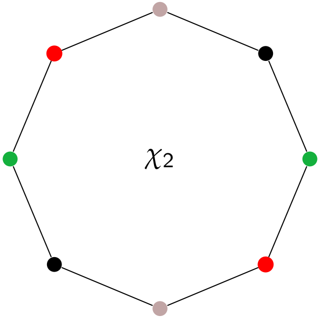

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k-chromatic number χ_k

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$$\chi_k(G) \geq \frac{|V(G)|}{\alpha_k(G)}$$

1 The origin of this project

SPCodingSchool

January 19th to 30th 2015 - Campinas, Brazil

S Coding and Information School

Note that ...

$$\alpha_k(G) = \alpha(G^k) \text{ and } \chi_k(G) = \chi(G^k)$$

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even the simplest spectral or combinatorial parameters of G^k cannot be always deduced from the parameters of G.

Examples:

- average degree (Devos, McDonald and Scheide 2013)
- rainbow number (Basavaraju, Chandran, Rajendraprasad and Ramaswamy 2014)
- eigenvalues
- ...

Motivation α_k

(Kong and Zhao 1993) Computing α_k and χ_k is NP-complete.

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$$\Downarrow$$

We will find bounds that only depend on the spectrum of G.

Goal

There are two classic eigenvalue bounds for $\alpha(G)$:

inertia bound

ratio bound

We will:

- Extend bounds to $\alpha_k(G)$ and $\chi_k(G)$ (in terms of eigenvalues of G)
- Optimize the new bounds using polynomials
- Find applications of the new bounds

Some known upper bounds on α_k

- (Firby and Haviland 1997) For connected graphs using average distance.
- (Fiol 1997) For regular graphs using eigenvalues and alternating polynomials.
- (Atkinson and Frieze 2003) For random graphs $G_{n,p}$, $p = d/n$ (d a large constant).
- (Beis, Duckworth and Zito 2005) For random r-regular graphs.
- (O, Shi and Taoqiu 2019) For r-regular graphs for every $k \geq 2$ and $r \geq 3$.
- (Jou, Lin and Lin 2020) For trees and $k = 2$.
- ...

Our main tool: interlacing

Let $m < n$.
Sequences $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$ **interlace** if

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \qquad (1 \leq i \leq m)$$

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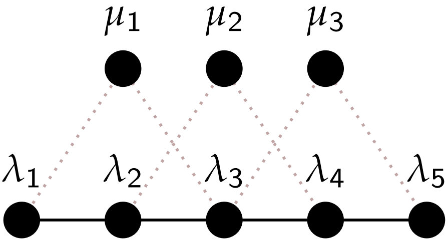

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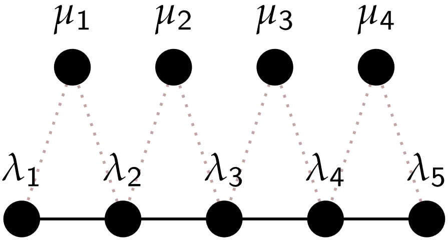

$m = n - 1$

Eigenvalue interlacing

$\lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of a matrix A

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$\mu_1, \mu_2, \ldots, \mu_m$ eigenvalues of a matrix B

First case of eigenvalue interlacing

1. B is a principal submatrix of A.

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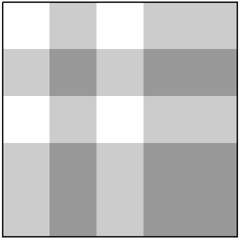
Theorem (Cauchy)
If B is a principal submatrix of A, then the eigenvalues of B interlace those of A.



Second case of eigenvalue interlacing

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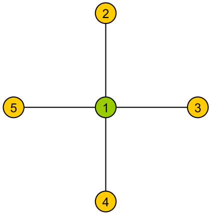


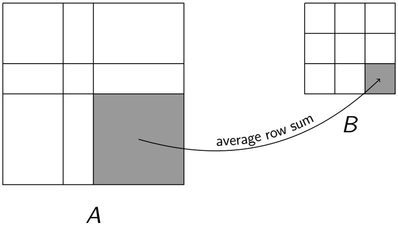
$$\mathcal{P} = \{\ V_1, V_2\ \} = \{\{\ 1\ \}, \{\ 2, 3, 4, 5\ \}\}$$

$$\overline{B} = (S^\top S)^{-1} S^\top A S = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

Second case of eigenvalue interlacing

2. If $\mathcal{P} = \{V_1, \ldots, V_m\}$ is a partition of V we can take for B the so-called **quotient matrix** of A with respect to $\mathcal{P}$.

Theorem (Haemers 1995)
If B is the quotient matrix of a partition of A, then the eigenvalues of B interlace the eigenvalues of A.



Classic eigenvalue bound I: inertia

Theorem (Inertia bound, Cvetković 1972)
If G is a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

Holds also more generally for weighted adjacency matrices.

Classic eigenvalue bound II: ratio

Theorem (Ratio bound, Hoffman 1970)
If G is regular with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}$$

and if an independent set C meets this bound then every vertex not in C is adjacent to precisely $-\lambda_n$ vertices of C.

Delsarte proved the ratio bound for SRGs, later Hoffman extended it to regular graphs and Haemers to irregular graphs

(Lovász 1979) showed that Lovász theta number $\vartheta(G)$ is a lower bound for the Hoffman bound.

More on the ratio bound

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Linear Algebra and its Applications

Volume 617, 15 May 2021, Pages 215-219

Hoffman's ratio bound

In memory of Alan J. Hoffman

Willem H. Haemers

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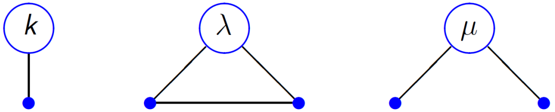
https://doi.org/10.1016/j.laa.2021.02.010

Abstract

Hoffman's ratio bound is an upper bound for the independence number of a regular graph in terms of the eigenvalues of the adjacency matrix. The bound has proved to be very useful and has been applied many times. Hoffman did not publish his result, and for a great number of users the emergence of Hoffman's bound is a black hole. With his note I hope to clarify the history of this bound and some of its generalizations.

Inertia vs ratio bound: SRG



Graph	(n,k,λ,μ)	α	Inertia bound	(Floor of) ratio bound
Cycle C_5	$(5,2,0,1)$	2	2	2
Petersen	$(10,3,0,1)$	4	4	4
Clebsh	$(16,5,0,2)$	5	**5**	**6**
Hoffman-Singleton	$(50,7,0,1)$	15	21	15
Gewirtz	$(56,10,0,2)$	16	20	16
Mesner M_{22}	$(77,16,0,7)$	21	21	21
Higman-Sims	$(100,22,0,6)$	22	**22**	**26**

Inertia vs ratio bound

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BUT the inertia bound can sometimes beat Lovász theta number!

E.g for C_5, inertia bound gives 2 while the Lovász theta number gives $\sqrt{5} = 2.2$.

$\downarrow$

It makes sense to investigate both the inertia and ratio bound.

Timeline

(1) (A., Cioabă and Tait 2016) New bounds on α_k in terms of λ_i^k.

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(1) and (2) do not consider the case when the spectra of G and G^k are related.

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Which polynomial gives the best bound for a specific graph?

(3) (A., Coutinho, Fiol, Nogueira, Zeijlemaker 2022) Optimize the bounds over $p \in \mathbb{R}_k[x]$.




Optimization and eigenvalue bounds

Independence number

- (Delsarte 1973) LP bound on α for distance-regular graphs.
- (Lovász 1979) SDP bound ϑ.
- ...

***k*-independence number**

Ratio-type bound	Inertial-type bound
$\downarrow$	$\downarrow$
(Fiol 2019) LP with minor polynomials	?

Plan

The spectrum of G^k and G are related

if there is a polynomial p s.t. $p(A(G)) = A(G^k)$, i.e., $A(G^k)$ belongs to the algebra generated by $A(G)$.

Graphs with large chromatic number

Question (Alon and Mohar 2000)

What is the largest possible value of the chromatic number $\chi(G^k)$ of G^k, among all graphs G with maximum degree at most d and girth at least g?

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What is the largest possible value of the chromatic number $\chi(G^k)$ of G^k, among all graphs G with maximum degree at most d and girth at least g?

- $k = 1$: long-standing problem by Vizing, settled asymptotically by (Johansson 1996) using the probabilistic method.
- $k = 2$: settled asymptotically by (Alon and Mohar 2002).
- $k \geq 3$: bounds by (Alon and Mohar 2002), (Kang and Pirot 2016), (Kang and Pirot 2018), ...

Lower bounds on χ_k

Let $G = (V, E)$ be a graph with spectrum $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$ and consider the inner product

$$\langle f, g \rangle_G = \frac{1}{n} \operatorname{tr}(f(A)g(A)) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\theta_i) g(\theta_i).$$

The **predistance polynomials** $p_0, \ldots, p_d$ are orthogonal polynomials with respect to the above product, with $\deg p_i = i$, and normalized such that $\|p_i\|_G^2 = p_i(\theta_0)$ (Fiol and Garriga 1997).

Lower bounds on χ_k

Let $G = (V, E)$ be a graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and predistance polynomials $p_0, \ldots, p_d$. For a given integer $k \leq d$, consider the polynomial $q_k = p_0 + \cdots + p_k$.

Lower bounds on χ_k

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Lemma (Fiol 2012)

Let $s_k(u)$ be the number of vertices at distance at most k from u. Then $q_k(\lambda_1)$ is bounded above by

$$q_k(\lambda_1) \leq H_k = \frac{n}{\sum_{i \in V} \frac{1}{s_k(u)}}.$$

Equality occurs if and only if $q_k(A) = I + A(G^k)$.

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Theorem (A., Coutinho, Fiol, Nogueira, Zeijlemaker 2022)
Let $q_k' = q_k - 1$. If G is regular with eigenvalues satisfying $q_k(\lambda_1) = H_k$, then

$$\chi_k(G) \geq \frac{n}{\min\{|\{i : q_k'(\lambda_i) \geq 0\}|, |\{i : q_k'(\lambda_i) \leq 0\}|\}}$$

and

$$\chi_k(G) \geq \frac{n}{1 - \frac{q_k'(\lambda_1)}{\min\{q_k'(\lambda_i)\}}}.$$

Lower bounds on χ_k: q_k?

First spectral bounds . . .

But how do we find the polynomial $q_k = p_0 + \cdots + p_k$?

Lower bounds on χ_k: q_k?

First spectral bounds ...

But how do we find the polynomial $q_k = p_0 + \cdots + p_k$?

Theorem (A., Van Dam and Fiol 2016)
It holds that $q_k(A) = A(G^k) + I$ when G is a δ-regular graph with girth g and $k = \lfloor \frac{g-1}{2} \rfloor$. In this case G is k-partially distance-regular, and

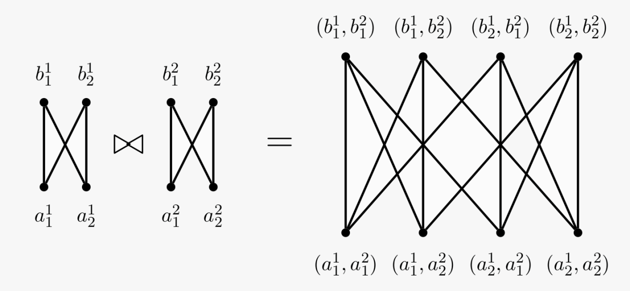
$$q_0 = 1,\ q_1 = 1 + x,\ q_{i+1} = xq_i - (\delta - 1)q_{i-1}.$$

Lower bounds on χ_k: Sage named graphs

Name	Girth g	$k = \lfloor \frac{g-1}{2} \rfloor$	α_k
Moebius-Kantor graph	6	2	4
Nauru graph	6	2	6
Blanusa First Snark graph	5	2	4
Blanusa Second Snark graph	5	2	4
Brinkmann graph	5	2	3
Heawood graph	6	2	2
Sylvester graph	5	2	6
Coxeter graph	7	3	4
Dyck graph	6	2	8
F26A graph	6	2	6
Flower Snark graph	5	2	5

Lower bounds on χ_k: tight families

(Kang and Pirot 2016) used **balanced bipartite products** $\bowtie$ for their lower bound construction.



This product also gives several graphs which attain equality for our bound, for example the products of even cycles $C_8 \bowtie C_8$, $C_8 \bowtie C_{12}$, ...

Plan

Inertial-type bound

Let G be a graph with adjacency matrix A and $p \in \mathbb{R}_k[x]$.

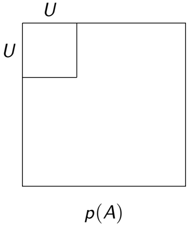
$$w(p) := \min_i \; p(A)_{ii}$$
$$W(p) := \max_i \; p(A)_{ii}$$

Theorem (A., Coutinho, Fiol 2019)
Let $p \in \mathbb{R}_k[x]$, then

$$\alpha_k(G) \le \min\{|i : p(\lambda_i) \ge w(p)|, |i : p(\lambda_i) \le W(p)|\}.$$

Proof idea

Let U be a k-independent set of G with size α_k.



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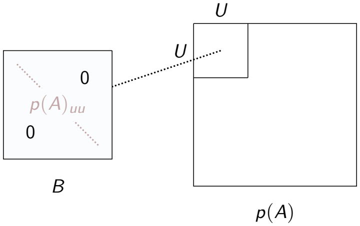

Let μ be the smallest eigenvalue of B.

- Cauchy interlacing ($\lambda_i \geq \mu_i$ for $i = 1, \ldots, m = |U|$): $\geq |U|$ eigenvalues of $p(A)$ are larger than μ
- $\mu \geq w(p)$ by definition of $w(p) = \min_{u \in V}\{(p(A))_{uu}$.

Therefore, $|U| \leq |\{i : p(\lambda_i) \geq w(p)\}|$.

Inertial-type bound: corollary

For $k = 1$,

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Corollary (Inertia bound, Cvetković 1972)
If G is a graph, then

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

Inertial-type bound: optimization (MILP)

$$\alpha_k(G) \leq \min\{|i : p_k(\lambda_i) \geq w(p_k)|, |i : p_k(\lambda_i) \leq W(p_k)|\}$$

Linear?

Inertial-type bound: optimization (MILP)

$$\alpha_k(G) \leq \min\{|i : p_k(\lambda_i) \geq w(p_k)|, |i : p_k(\lambda_i) \leq W(p_k)|\}$$

Linear?

Invariant under scaling and translation

- may assume $\min\{|i : p_k(\lambda_i) \geq w(p_k)|\}$, otherwise take $-p_k$
- translate: $\min\{|i : p_k(\lambda_i) \geq 0|\}$.

Inertial-type MILP: $\alpha_k \leq \min\{|i : p_k(\lambda_i) \geq 0|\}$

Let G be a graph with spectrum $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$ and $p_k(x) = a_k x^k + \cdots + a_0$ the polynomial to optimize.

For each $u \in V$, assume $w(p_k) = p_k(A)_{uu}$ (i.e. u is the smallest entry) and solve

$$
\begin{array}{rl}
\texttt{minimize} & \boldsymbol{m}^T\boldsymbol{b} \\
\texttt{subject to} & \sum_{i=0}^{k} a_i (A^i)_{vv} \geq 0, \quad v \in V(G)\backslash\{u\} \\
& \sum_{i=0}^{k} a_i (A^i)_{uu} = 0 \\
& \sum_{i=0}^{k} a_i {\theta_j}^i - Mb_j + \varepsilon \leq 0, \quad j = 0, \ldots, d \\
& \boldsymbol{b} \in \{0,1\}^{d+1}
\end{array}
$$

with M large, $\varepsilon > 0$ small.

Input size: $d + 1$

Inertial-type MILP: $\alpha_k \leq \min\{|i : p_k(\lambda_i) \geq 0|\}$

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Vector b encodes whether $p_k(\theta_i) \geq w(p_k)$: $b_i = 1$ iff $p_k(\theta_i) \geq 0$ (if $p_k(\theta_i) = 0$ then we need ε to force $b_i = 1$)

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$$

with M large, $\varepsilon > 0$ small.

Linear combination of the eigenvalues mutiplicities (minimizing the quantity of indices j): $\min \sum_{j : p(\theta_j) \geq w(p) = 0} m_j$

Inertial-type MILP: results

For large graphs, solving n MILPs takes a lot of time. However, this first inertial-type bound does not require walk-regularity as the ratio-type bound optimized by (Fiol 2020).

Proportion of small irregular graphs for which the optimal solution of the MILP equals α_2:

Number of vertices	4	5	6	7	8	9
Proportion	0.86	0.84	0.76	0.62	0.46	0.27

Inertial-type MILP: Sage named graphs

Name	Best 2019	ϑ_2	First MILP	α_2
Balaban 10-cage	17	17	19	17
Frucht graph	3	3	3	3
Meredith Graph	14	10	10	10
Moebius-Kantor Graph	4	4	6	4
Bidiakis cube	3	2	4	2
Gosset Graph	2	2	8	2
Gray graph	14	11	19	11
Nauru Graph	6	5	8	6
Blanusa First Snark Graph	4	4	4	4
Pappus Graph	4	3	7	3
Blanusa Second Snark Graph	4	4	4	4
Poussin Graph	-	2	4	2
Brinkmann graph	4	3	6	3
Harborth Graph	12	9	13	10
Perkel Graph	10	5	18	5
Harries Graph	17	17	18	17
Bucky Ball	16	12	16	12
Harries-Wong graph	17	17	18	17
Robertson Graph	3	3	5	3
Heawood graph	3	2	2	2
Herschel graph	-	2	3	2
Hoffman Graph	3	2	5	2
...				

Inertial-type MILP: walk-regular graphs

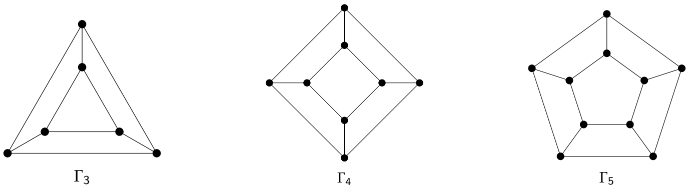
Let G be a k-partially walk-regular. Then $p_k(A)$ has constant diagonal (all vertices have the same number of closed walks of length smaller of equal than k), so we only have to run the MILP once:

$$
\begin{array}{rl}
\texttt{minimize} & \boldsymbol{m}^T \boldsymbol{b} \\
\texttt{subject to} & \sout{\sum_{i=0}^{k} a_i (A^i)_{vv} \geq 0, \quad v \in V(G) \backslash \{u\}} \\
 & \sout{\sum_{i=0}^{k} a_i (A^i)_{uu} = 0} \\
 & \sum_{i=0}^{d} m_i p_k(\theta_i) = 0 \\
 & \sum_{i=0}^{k} a_i {\theta_j}^i - M b_j + \varepsilon \leq 0, \quad j = 0, \ldots, d \\
 & \boldsymbol{b} \in \{0,1\}^{d+1}
\end{array}
$$

same condition as for the ratio-type bound optimized by (Fiol 2020)

Inertial-type MILP: equality $k = 2$

Prism graphs Γ_n

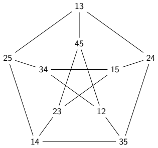


$$\alpha_2(\Gamma_{4i+j}) = \begin{cases} 2i+1 & \text{if } j = 3 \\ 2i & \text{otherwise} \end{cases}$$

These graphs are walk-regular. For $n \neq 2 \mod 4$, the MILP is tight.

Inertial-type MILP: equality

Odd graph O_ℓ: vertices corresponding to the $(\ell-1)$-subsets of a $(2\ell-1)$-set, and the adjacencies are defined by void intersection.



O_3

Corollary (A., Coutinho, Fiol, Nogueira, Zeijlemaker 2022)
For $i=0,\ldots,\ell-1$, let μ_i and m_i be the eigenvalues and multiplicities of the Odd graph $O_\ell = O_{d+1}$. Then,

$$\alpha_{d-1}(O_{d+1}) \leq \left\{ \begin{array}{ll} m_1 & \text{for even } d \\ m_1+1 & \text{for odd } d \end{array} \right\} = \left\{ \begin{array}{ll} 2d & \text{for even } d, \\ 2d+1 & \text{for odd } d. \end{array} \right.$$

When is the previous bound tight?

Odd graph $O_{d+1} (= O_\ell)$	α_{d-1}	Inertial-type bound
O_2 (K_3)	$\alpha_0 = 3$	$m_0 + m_1 = 3$
O_3 (Petersen)	$\alpha_1 = 4$	$m_1 = 4$
O_4	$\alpha_2 = 7$	$m_0 + m_1 = 7$
O_5	$\alpha_3 = 7$	$m_1 = 8$
O_6	$\alpha_4 = 11$	$m_0 + m_1 = 11$
O_7	$\alpha_5 = 12$	$m_1 = 12$
O_8	$\alpha_6 = 15$	$m_0 + m_1 = 15$
O_9	$\alpha_7 = 15$	$m_1 = 16$
O_{10}	$\alpha_8 = 19$	$m_0 + m_1 = 19$
O_{11}	$\alpha_9 = 19$	$m_1 = 20$
O_{12}	$\alpha_{10} = 23$	$m_0 + m_1 = 23$
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when ℓ is even, the exact value of α_{d-1} can be proved theoretically through its relation with symmetric designs: $\alpha_{\ell-2}(O_\ell) = 2\ell - 1$ ($\alpha_{d-1}(O_{d+1}) = 2d + 1$), if and only if the vertices of a maximum $(\ell - 2)$-independent set constitute a $2 - (2\ell - 1, \ell - 1, \frac{1}{2}\ell - 1)$ symmetric design $\Rightarrow$ such designs are known to exist for $\ell = 4, 6, \ldots, 16$ (Stinson's book)

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when ℓ is odd, values found by computer ...

Plan

Ratio-type bound

$W(p) := \max_{u \in V}\{(p(A))_{uu}\}$
$w(p) := \min_{u \in V}\{(p(A))_{uu}\}$
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Theorem (A., Coutinho, Fiol 2019)
Let G be a regular graph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$ and $\lambda(p)$, and assume $p(\lambda_1) > \lambda(p)$. Then,

$$\alpha_k(G) \leq n\frac{W(p) - \lambda(p)}{p(\lambda_1) - \lambda(p)}.$$

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$$\chi_k(G) \geq \frac{p(\lambda_1) - \lambda(p)}{W(p) - \lambda(p)}.$$

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$\lambda(p) := \max_{i\in[2,n]}\{p(\lambda_i)\}$

Theorem (A., Coutinho, Fiol, Nogueira, Zeijlemaker 2022)
Let G be a ~~regular~~ graph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$ and $\lambda(p)$, and assume $p(\lambda_1) > p(\lambda_i)$ for $i = 2, \ldots, n$. Then,

$$\chi_k(G) \geq \frac{p(\lambda_1) - \lambda(p)}{W(p) - \lambda(p)}.$$

Proof ingredients: quotient matrix interlacing (+ weight-equitable partitions)

Ratio-type bound: corollary

For $k = 1$,

Ratio-type bound: corollary

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Corollary (Ratio bound, Hoffman 1970)

If G is regular then

$$\alpha(G) \leq n\frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

Ratio-type bound: best polynomial for $k = 2$

Theorem (A., Coutinho, Fiol 2019)
Let G be a δ-regular graph with n vertices and distinct eigenvalues $\theta_0(=\delta) > \theta_1 > \cdots > \theta_d$ with $d \geq 2$. Let θ_i be the largest eigenvalue such that $\theta_i \leq -1$. Then,

$$\alpha_2(G) \leq n \frac{\theta_0 + \theta_i \theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}.$$

Moreover, this is the best possible bound that can be obtained by choosing a polynomial and applying the ratio-type bound.

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Ratio-type bound: best polynomial for $k = 3$

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Discrete Mathematics
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The optimal bound on the 3-independence number obtainable from a polynomial-type method

Lord C. Kavi, Mike Newman

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Abstract

A k-independent set in a connected graph is a set of vertices such that any two vertices in the set are at distance greater than k in the graph. The k-independence number of a graph, denoted α_k, is the size of a largest k-independent set in the graph. Recent results have made use of polynomials that depend on the spectrum of the graph to bound the k-independence number. They are optimized for the cases $k = 1, 2$. There are polynomials that give good (and sometimes) optimal results for general k, including case $k = 3$. In this paper, we provide the best possible bound that can be obtained by choosing a polynomial for case $k = 3$ and apply this bound to well-known families of graphs including the Hamming graph.

Ratio-type bound: best polynomial for $k = 3$

Theorem (Neuwman and Kavi 2023)
Let G be a d-regular graph with n vertices, adjacency matrix A, and distinct eigenvalues $d = \theta_0 > \theta_1 > \cdots > \theta_d$, with $d \geq 3$. Let s be the largest index such that $\theta_s \geq -\frac{\theta_0^2+\theta_0\theta_d-\Delta}{\theta_0(\theta_d+1)}$, where $\Delta = \max_{u\in V}\{(A^3)_{uu}\}$. Let $b = -(\theta_s + \theta_{s+1} + \theta_d)$ and $c = \theta_d\theta_s + \theta_d\theta_s + 1 + \theta_s\theta_{s+1}$. Then, $p(x) = x^3 + bx^2 + cx$ is an optimal polynomial for $k = 3$.
The corresponding bound on the 3-independence number of G is

$$\alpha_3(G) \leq n\frac{\Delta - \theta_0(\theta_s + \theta_{s+1} + \theta_d) - \theta_s\theta_{s+1}\theta_d}{(\theta_0 - \theta_s)(\theta_0 - \theta_{s+1})(\theta_0 - \theta_d)}.$$

Ratio-type bound: best polynomial for $k = 3$

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Ratio-type bound: optimization (LP)

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A new class of polynomials from the spectrum of a graph, and its application to bound the k-independence number

Check for updates

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Applicable to k-partially walk-regular graphs:

$$(p(A))_{uu} = \frac{1}{n}\operatorname{tr} p(A) = \frac{1}{n}\sum_{i=1}^{n} p(\lambda_i) \quad \text{for all } u \in V.$$

while our MILP for the inertial-type bound is for general graphs.

For free: distance-t chromatic index

distance between two vertices: number of edges in a shortest path in G between them.

distance between two edges: number of vertices in a shortest path between them. Incident edges have distance 1.

For free: distance-t chromatic index

- **strong edge-colouring**: colouring of the edges such that no two edges within distance 2 are given the same colour.
- **strong chromatic index** $\chi'_2(G)$: the least integer k such that there exists a strong edge-colouring of G using k colours.

For free: distance-t chromatic index

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(Kang, Manggala 2012) proposed a distance-based generalization of $\chi_2'(G)$:

- **distance-t chromatic index** $\chi_t'(G)$: smallest number of colours required to colour all the edges of G, such that no two edges within distance t share a colour. Probabilistic bounds by (Kang, Manggala 2012), (Kaiser, Kang 2014), (Cambie 2022), ...

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Using $\chi_t(G) \geq \frac{|V|}{\alpha_t(G)}$ and $\chi'_t(G) = \chi_t(L(G)) \rightarrow$ sharp eigenvalue bounds for $\chi'_t(G)$

Plan

Applications MILPs

Why optimizing the spectral bounds using MILPs?

Applications MILPs

Why optimizing the spectral bounds using MILPs?

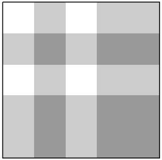
Applications:

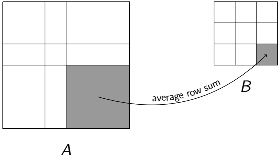
Applications MILPs

Why optimizing the spectral bounds using MILPs?

Applications:

- towards unifying eigenvalue interlacing





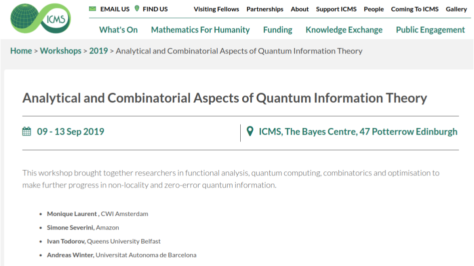
minor polynomials (LP)

Applications MILPs

Why optimizing the spectral bounds using MILPs?

Applications:

- towards unifying eigenvalue interlacing
- quantum information theory



Applications MILPs

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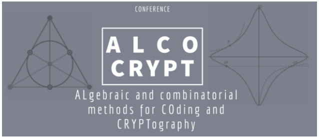
- coding theory



Towards unifying eigenvalue interlacing

While both the inertia and ratio type bound give proofs of the famous EKR, their relationship (and the relation of the interlacing types both use) is not well understood.

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We use the polynomials involved in the optimization of:

- inertial-type bound → sign polynomials (MILP) (A., Coutinho, Fiol, Nogueira, Zeijlemaker 2022)
- ratio-type bound → minor polynomials (LP) (Fiol 2020)

to find relationships between the bounds (A., Dalfó, Fiol, Zeijlemaker 2023) → via **(tight)** k-**CH-graphs**

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Tight 1-CH strongly regular graphs investigated by (Haemers and Higman 1989)

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The Odd graph O_ℓ with even degree ℓ is a $(d-1)$-CH graph, and O_ℓ is a tight $(d-1)$-CH graph for $\ell \in \{2, 3, 4, 6, 7, 8, 10, 12, 14, 16\}$.

Quantum independence number α_q

- (Mancinska and Roberson 2016) viewed it in terms of quantum homomorphisms, not known whether α_q is computable
- (Scarpa 2013) infinite families with $\alpha < \alpha_q$
- (Piovesan 2016) smallest graph on 24 vertices with $\alpha < \alpha_q$
- (Godsil and Sobchuk 2022) a characterization of when $\alpha < \alpha_q$

Quantum independence number α_q

The largest t such that there exists a $V(G) \times t$ matrix P with entries $P_{ui} \in \mathbb{C}^{d \times d}$ projections such that:

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(in every column the sum of projections has to be the identity $\Rightarrow$ those projections are orthogonal)

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(2) $P_{ui}P_{vj} = 0 \quad \forall uv \in E(G)$ for all $i, j = 1, \dots, t$

(two projections in different columns i,j and different rows u,v: if there is an edge the projections have to be orthogonal → related to the classical α*)*

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(in the rows, the projections are orthogonal)

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We call P a **quantum coclique**.

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We call P a **quantum coclique**.

The **quantum independence number** α_q is the largest number of colors that we an have in a quantum coclique.

$\alpha \leq \alpha_q$

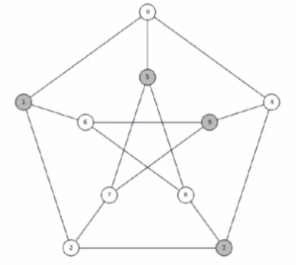

Figure: classical 4-coclique

$$\begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

entries=projections dim=1

standard vector in each row/vertex of the coclique

(1) sum in every col is 1

(2) adjacency $\rightarrow$ orthogonal

(3) rows are orthogonal

For the separation to occur, dim > 1

Bound on α_{kq}

The quantum k-independence number is upper bounded by the inertial-type bound (A., Elphick and Wocjan 2022):

$$\alpha_k(G) \leq \alpha_{kq}(G) \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

NOT known whether the quantum parameter $\alpha_{kq}(G)$ is computable!

- if inertial bound is saturated $\rightarrow \alpha_k = \alpha_{kq}$
- else $\rightarrow$ separation $\alpha_k < \alpha_{kq}$

For $k > 1$ and when the inertial-type bound is tight, we can use its MILP from (A., Coutinho, Fiol, Nogueira, Zeijlemaker 2022) to compute the value of the quantum parameter α_{kq}.

Coding theory

- A field $\mathbb{F}$ (usually $\mathbb{F}_q$)
- V finite dimensional $\mathbb{F}_q$-vector space
- $\mathrm{wt} : V \to \mathbb{N}$ **weight function**
- $\delta : V \times V \to \mathbb{R}_{\geq}$; $\delta(u, v) := wt(u - v)$ **distance function**
- (V, δ) **metric space**

A $[V, k, d]$ **code** $\mathcal{C}$ is a k-dimensional $\mathbb{F}_q$-subspace of V endowed with the metric δ. The parameter d is known as the minimum distance of $\mathcal{C}$, that is

$$d = \min\{\delta(u, v) : u, v \in \mathcal{C}, u \neq v\} = \min\{\mathrm{wt}(u) : u \in \mathcal{C} \setminus 0\}$$

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Question

What is the maximal size of a [insert metric] code with minimum distance d? Bounds for $A_q(k, d)$?

Sum-rank metric codes

- The **sum-rank-metric space**:

  $(\mathbb{F}_q^{m\times n})^t = \mathbb{F}_q^{m\times n} \times \cdots \times \mathbb{F}_q^{m\times n} \leftarrow$ t tuples of matrices of size $n \times m$

- The **sum-rank (weight)** of $(X_1, \ldots, X_t) \in (\mathbb{F}_q^{m\times n})^t$:
  $\mathrm{wt}_{srk}(X) = \sum_{i=1}^{t} \mathrm{rk}(X_i)$

- The **sum-rank distance** between $X, Y \in (\mathbb{F}_q^{m\times n})^t$:
  $d_{srk}(X, Y) = \mathrm{wt}_{srk}(X - Y) = \sum_{i=1}^{t} \mathrm{rk}(X_i - Y_i)$

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Example

$$X = (X_1, X_2) = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \in (\mathbb{F}_2^{3\times 2})^2$$

$$wt_{srk}(X) = \mathrm{rk}(X_1) + \mathrm{rk}(X_2) = 3$$

Sum-rank metric codes

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Question

What is the maximal size of a sum-rank metric code with minimum distance d? Bounds for $A_q(k, d)$?

Eigenvalue bounds for srk and Lee codes

Step 1 geodesic distance between two vertices in the graph associated to [the metric] = [the metric] distance

Step 2 $(d-1)$-independence number α_{d-1} of a graph associated to [the metric] = the maximal size of a code in [the metric] with minimum distance d

Ideally: graphical metric with a non distance-regular graph (else Delsarte LP bound applies)

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Example

sum-rank metric, Lee metric, ...

Eigenvalue bounds for srk and Lee codes

New result (A., Khramova, Ravagnani 2023++)
Ratio-type bound for the cardinality $A_q(k, d)$ of sum-rank metric codes.

For some instances this outperforms the best known coding theory bounds by (Byrne, Gluesing-Luerssen, Ravagnani 2021).

Details at 16h in Khramova's contributed talk.

Eigenvalue bounds for srk and Lee codes

New result (A., Rijnders 2023++)
Ratio-type bound for the cardinality $A_q(k, d)$ of Lee codes.

Equally good as some known sharp combinatorial bounds by (Kim and Kim 2011), ...

Plan

Overview results

Optimization independence number

- (Delsarte 1973) LP bound on α for distance-regular graphs.
- (Lovász 1979) SDP bound ϑ.
- ...

Optimization k-independence number

Ratio-type bound	Inertial-type bound
↓	↓
(Fiol 2020) LP with minor polynomials	**this talk** MILP with sign polynomials

Open problems

- Complexity of the MILPs? Does increasing k make the problem easier?
- Use the MILPs for other graphs and values of k, and find more closed formulas for graph families.
- SDP formulation for the inertial-type bound?

$$\alpha \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}$$
$$\alpha_k \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}$$

- Characterize more cases of k-CH-graphs.
- Other bounds on α_{kq}?

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Thank you for listening!