Motzkin Numbers and Flag Codes

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Joint work with Miguel Ángel Navarro-Pérez

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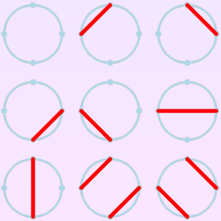
Motzkin numbers

The Motzkin numbers sequence $\{M_n\}_{n=0}^{\infty}$, whose first ten terms are

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835,$$

was introduced by T. Motzkin, while counting possible sets of nonintersecting chords joining some of n points on a circle.



$M_4 = 9$

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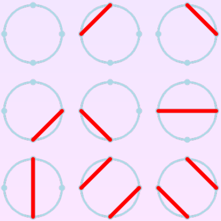

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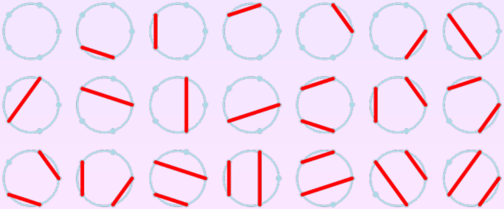

$$M_5 = 21$$

Motzkin numbers

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A001006 Motzkin numbers: number of ways of drawing any number of nonintersecting chords joining n (labeled) points on a circle. 525
(Formerly M1184 N0456)

1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, 2356779, 6536382, 18199284, 50852019, 142547559, 400763223, 1129760415, 3192727797, 9043402501, 25669818476, 73007772802, 208023278209, 593742784829 (list; graph; refs; listen; history; text; internal format)

Motzkin numbers

We can compute $M_0 = 1$ and

$$M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-k-2}.$$

There are many different combinatorial objects counted by this sequence. We are interested in Motzkin paths.

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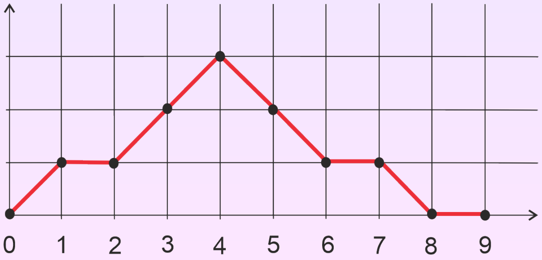
Motzkin paths

Definition

A *Motzkin path* of length n is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(n,0)$ that never runs below the x-axis and whose permitted steps are the up diagonal step $(1,1)$, the down diagonal step $(1,-1)$, and the horizontal step $(1,0)$.



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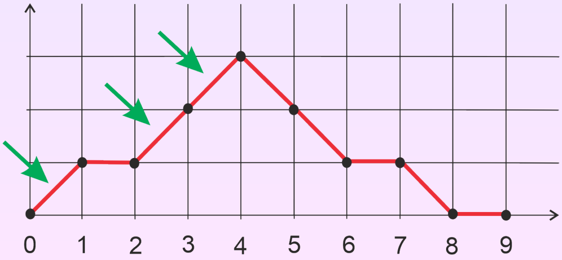

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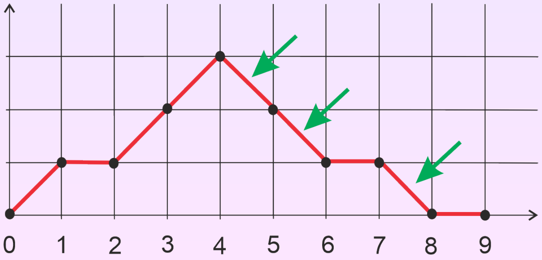

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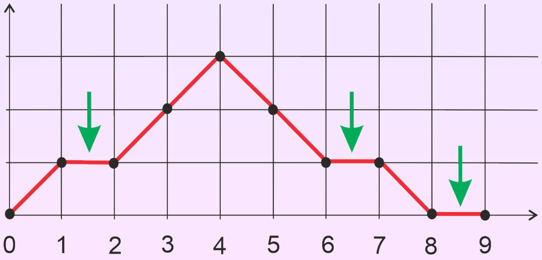

Motzkin paths

The number of Motzkin paths of length n is given by the Motzkin number M_n.

Notation

We consider:

- q a prime power and $n \geq 2$ a positive integer,
- $\mathbb{F}_q$ the finite field with q elements,
- $\mathbb{F}_q^n$ the n-dimensional vector space over $\mathbb{F}_q$,
- $\mathcal{G}_q(k, n)$, the Grassmannian of dimension k for $1 \leq k \leq n$.

Constant dimension codes

Given subspaces $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(k, n)$, their injection distance is

$$d_I(\mathcal{U}, \mathcal{V}) = k - \dim(\mathcal{U} \cap \mathcal{V}).$$

Definition

A constant dimension code $\mathcal{C}$ in $\mathcal{G}_q(k, n)$ is a nonempty set of k-dimensional subspaces of $\mathbb{F}_q^n$. Its minimum (injection) distance is

$$d_I(\mathcal{C}) = \min\{d_I(\mathcal{U}, \mathcal{V}) \,|\, \mathcal{U}, \mathcal{V} \in \mathcal{C},\ \mathcal{U} \neq \mathcal{V}\}$$

and it takes values in $\{0, 1, \dots, \min\{k, n-k\}\}$.

Introduced by Koetter and Kschischang in 2008.

Flags

Definition

A flag of length r on $\mathbb{F}_q^n$ is a sequence

$$\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$$

of $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$ satisfying

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_r \subsetneq \mathbb{F}_q^n.$$

The increasing sequence of dimensions

$$(\dim_q(\mathcal{F}_1), \ldots, \dim_q(\mathcal{F}_r))$$

is called the type of $\mathcal{F}$. If it is $(1, 2, \ldots, n-1)$, $\mathcal{F}$ is a full flag.

Flag distance

Given flags $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_r)$ of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$, their flag distance is

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_I(\mathcal{F}_i, \mathcal{F}'_i).$$

Flag codes

Definition

A flag code $\mathcal{C}$ on $\mathbb{F}_q^n$ is a nonempty collection of flags of the same type. Its minimum distance is

$$d_f(\mathcal{C}) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'\}.$$

Introduced by *Liebhold, Nebe and Vázquez-Castro* in 2018.

Distance vectors: how to spread the flag distance

We always have

$$0 \leq d_f(\mathcal{C}) \leq D^{(t,n)} = \left(\sum_{t_i \leq \lfloor \frac{n}{2} \rfloor} t_i + \sum_{t_i > \lfloor \frac{n}{2} \rfloor} (n - t_i) \right).$$

When working with full flags,

$$0 \leq d_f(\mathcal{C}) \leq D^{n} = \left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even,} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Projected codes of a flag code

$\mathcal{C}$ a flag code $\rightsquigarrow$ associated constant dimension codes

Definition

Let $\mathcal{C}$ be a flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. For every index $i \in \{1, \ldots, r\}$, the *i*-projected code of $\mathcal{C}$ is the set of i-th subspaces $\mathcal{C}_i = p_i(\mathcal{C})$ where

$$p_i : \mathcal{G}_q(t_1, n) \times \cdots \times \mathcal{G}_q(t_r, n) \to \mathcal{G}_q(t_i, n)$$

is the i-th projection.

Question

Relationship between...
a flag code $\mathcal{C}$ and its projected codes $\mathcal{C}_i \subseteq \mathcal{G}_q(t_i, n)$

- Concerning the size: $|\mathcal{C}_i| \leq |\mathcal{C}|$.
- Concerning the distance: it could happen that

$$d_I(\mathcal{C}_i) > d_f(\mathcal{C}), \quad d_I(\mathcal{C}_i) < d_f(\mathcal{C}), \quad d_I(\mathcal{C}_i) = d_f(\mathcal{C}).$$

Main difficulty: A flag distance value can be obtained as a sum from different combinations

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Main difficulty: A flag distance value can be obtained as a sum from different combinations

Question

Example

Let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the standard basis of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^6$. Consider the flag code $\mathcal{C}$ of type $(1, 3, 5)$ on $\mathbb{F}_q^6$ given by the set of flags:

$$\begin{aligned}
\mathcal{F}^1 &= (\langle e_1 \rangle, \langle e_1, e_2, e_3 \rangle, \langle e_1, e_2, e_3, e_4, e_5 \rangle), \\
\mathcal{F}^2 &= (\langle e_5 \rangle, \langle e_4, e_5, e_6 \rangle, \langle e_1, e_2, e_4, e_5, e_6 \rangle), \\
\mathcal{F}^3 &= (\langle e_6 \rangle, \langle e_4, e_5, e_6 \rangle, \langle e_2, e_3, e_4, e_5, e_6 \rangle), \\
\mathcal{F}^4 &= (\langle e_2 \rangle, \langle e_2, e_5, e_6 \rangle, \langle e_2, e_3, e_4, e_5, e_6 \rangle).
\end{aligned}$$

Observe that it holds:

$$d_f(\mathcal{F}^2, \mathcal{F}^3) = 1 + 0 + 1 = 2 = 1 + 1 + 0 = d_f(\mathcal{F}^3, \mathcal{F}^4).$$

Distance vectors: how to spread the flag distance

To totally capture the relative position of two flags, we need to provide more precise information beyond the distance value.

Definition

Given two different flags $\mathcal{F}, \mathcal{F}'$ of type $t = (t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$, their associated *distance vector* is

$$\mathbf{d}(\mathcal{F}, \mathcal{F}') = (d_I(\mathcal{F}_1, \mathcal{F}'_1), \ldots, d_I(\mathcal{F}_r, \mathcal{F}'_r)) \in \mathbb{Z}^r.$$

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Distance vectors: how to spread the flag distance

Question

How many possible distance vectors could correspond to a given couple of arbitrary full flags $\mathcal{F}, \mathcal{F}'$ on $\mathbb{F}_q^n$, that is, what is the cardinality of the set

$$\mathcal{D}(n) = \{\mathbf{d}(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \text{ full flags on } \mathbb{F}_q^n\} \subseteq \mathbb{Z}^{n-1}.$$

Allowed distance combinations

Theorem (A-G. and Navarro-Pérez, 2022)

Put $\delta_0 = \delta_n = 0$ and consider integers $\delta_1, \ldots, \delta_{n-1} \geq 0$. Then, there exists a couple of full flags $\mathcal{F}$, $\mathcal{F}'$ such that

$$d_I(\mathcal{F}_i, \mathcal{F}'_i) = \delta_i$$

if, and only if,

$$\delta_i - \delta_{i-1} \in \{-1, 0, 1\},$$

for all $1 \leq i \leq n$.

The bijection

Theorem (A-G. and Navarro-Pérez, 2022)

Given $n \geq 2$, there is a bijection Ψ between the set of possible distance vectors $\mathcal{D}(n)$ and the set of Motzkin paths $\mathcal{M}_n$.

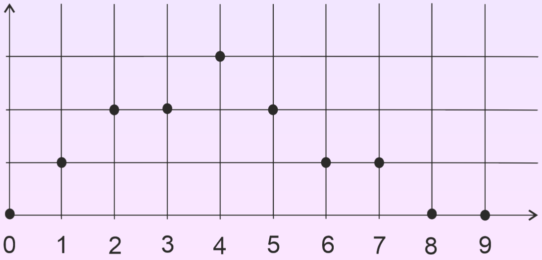
Proof:

$$\begin{array}{ccc} \mathcal{D}(n) & \xrightarrow{\Psi} & \mathcal{M}_n \\ (\delta_1, \dots, \delta_{n-1}) & \longmapsto & \{(0, \delta_0), (1, \delta_1), \dots, (n-1, \delta_{n-1}), (n, \delta_n)\}. \end{array}$$

Graphically...

Given full flags $\mathcal{F}$, $\mathcal{F}'$ with $d_I(\mathcal{F}_i, \mathcal{F}'_i) = \delta_i$ and fix $\delta_0 = \delta_n = 0$.

- Plot the points (i, δ_i) in $\mathbb{Z}^2$ for $0 \leq i \leq n$.
- Match them to draw the Motzkin path $\mathcal{M}_{\mathcal{F},\mathcal{F}'}$.



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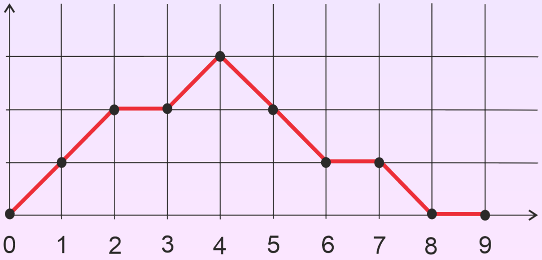

The bijection

Consequence

- Distance vectors give Motzkin paths and conversely.
- The number of possible flag distance combinations for full flags on $\mathbb{F}_q^n$ is exactly the n-th Motzkin number.

This is a new appearence of the Motzkin numbers sequence!

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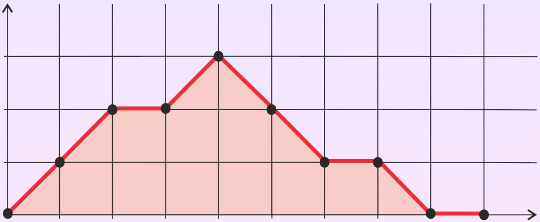
The bijection

Important remark!

The area of the region determined by $\mathcal{M}_{\mathcal{F},\mathcal{F}'}$ and the abscisa axis is equal to the flag distance $d_f(\mathcal{F},\mathcal{F}')$...



The area under $\mathcal{M}_{\mathcal{F},\mathcal{F}'}$ is 12.

Distance vectors of a flag code

Definition

Given a full flag code $\mathcal{C}$ on $\mathbb{F}_q^n$, its set of distance vectors is defined as

$$\mathcal{D}(\mathcal{C}) = \{\mathbf{d}(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ d_f(\mathcal{F}, \mathcal{F}') = d_f(\mathcal{C})\}.$$

Distance vectors of a flag code

Example

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{F}_q^4$. Take $\mathcal{C}$ given by

$$\begin{array}{rcl} \mathcal{F}^1 & = & (\langle e_1\rangle, \quad \langle e_1, e_2\rangle, \quad \langle e_1, e_2, e_4\rangle), \\ \mathcal{F}^2 & = & (\langle e_1\rangle, \quad \langle e_1, e_3\rangle, \quad \langle e_1, e_2, e_3\rangle), \\ \mathcal{F}^3 & = & (\langle e_2\rangle, \quad \langle e_2, e_3\rangle, \quad \langle e_1, e_2, e_3\rangle). \end{array}$$

Notice that

$$\begin{array}{rcccccccc} d_f(\mathcal{F}^1, \mathcal{F}^2) & = & 0 & + & 1 & + & 1 & = & 2, \\ d_f(\mathcal{F}^1, \mathcal{F}^3) & = & 1 & + & 1 & + & 1 & = & 3, \\ d_f(\mathcal{F}^2, \mathcal{F}^3) & = & 1 & + & 1 & + & 0 & = & 2. \end{array}$$

Hence, $d_f(\mathcal{C}) = 2$ and

$$\mathcal{D}(\mathcal{C}) = \{(0, 1, 1), (1, 1, 0)\}.$$

Distance vectors of a flag code

Question

What is the maximum number of distance vectors associated with a full flag code $\mathcal{C}$ on $\mathbb{F}_q^n$ with prescribed minimum distance?

The Motzkin paths of a flag code

Definition

Given a full flag code $\mathcal{C}$ on $\mathbb{F}_q^n$, its *set of Motzkin paths* is defined as

$$\mathcal{M}(\mathcal{C}) = \Psi(\mathcal{D}(\mathcal{C})).$$

The Motzkin paths of a flag code

Equivalent question

What is the maximum number of Motzkin paths of a full flag code $\mathcal{C}$ on $\mathbb{F}_q^n$ with prescribed minimum distance?

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Search: **motzkin path area**

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Sort: relevance | references | number | modified | created Format: long | short | data

A129181 Triangle read by rows: T(n,k) is the number of **Motzkin paths** of length n such that the **area** between the x-axis and the **path** is k (n>=0; 0<=k<=floor(n^2/4)). +60 4

1, 1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 1, 4, 6, 4, 3, 2, 1, 1, 5, 10, 10, 8, 7, 5, 3, 1, 1, 1, 6, 15, 20, 19, 18, 16, 12, 8, 6, 3, 2, 1, 1, 7, 21, 35, 40, 41, 41, 36, 29, 23, 18, 12, 9, 5, 3, 1, 1, 1, 8, 28, 56, 76, 86, 93, 92, 83, 72, 62, 50, 40, 30, 22, 14, 10, 6, 3, 2, 1, 1, 9, 36, 84, 133, 168 (list; graph; refs; listen; history; text; internal format)

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The Motzkin paths of a flag code

Theorem (A-G. and Navarro-Pérez, 2022)

Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$ such that

$$|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_{n-1}|,$$

then $\mathcal{M}(\mathcal{C})$ only contains elevated Motzkin paths.

The converse is not true!

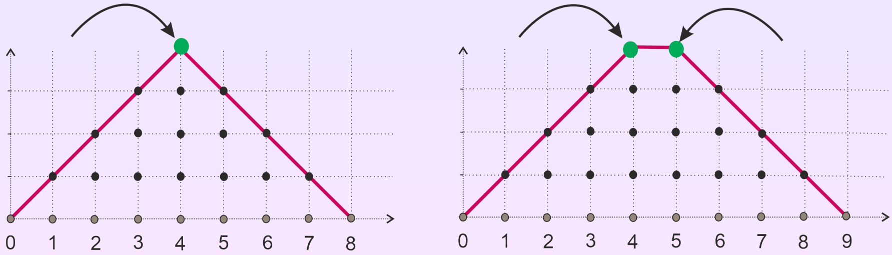
The Motzkin paths of a flag code

Theorem (A-G. and Navarro-Pérez, 2022)

Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$. They are equivalent:

1. $d_f(\mathcal{C}) = D^n$ ($\mathcal{C}$ is of maximum distance).
2. The set $\mathcal{M}(\mathcal{C})$ consists of the only Motzkin path passing either through the point $(\frac{n}{2}, \frac{n}{2})$, if n is even, or through the points $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)$ and $(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$, if n is odd.

The Motzkin paths of a flag code



The Motzkin paths of a flag code

Consequence:

Any ODFC $\mathcal{C}$ of full type is completely determined by just one or two of its projected codes. More precisely:

$\mathcal{C}$ is an ODFC $\Leftrightarrow$ $\mathcal{C}_{\frac{n}{2}}$ (n even) or $\mathcal{C}_{\lfloor \frac{n}{2} \rfloor}, \mathcal{C}_{\lceil \frac{n}{2} \rceil}$ (n odd) are maximum distance constant dimension codes with size $|\mathcal{C}|$.

Thank you very much for your attention!

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