Classification of Point-Primitive Generalised Quadrangles

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Euler's 36 Officers Problem

- 6 regiments (colours) each with a team of 6 officers
- 6 ranks (piece type)
- Each regiment has one officer of each rank
- Arrange officers in a 6×6 grid so that there are 6 officers of different ranks and different regiments

Example with 5 Officers

Figure 1: Taken from https://www.quantamagazine.org/eulers-243-year-old-impossible-puzzle-gets-a-quantum-solution-20220110/

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- It's impossible!

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- A certain finite geometric object doesn't exist

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- Finite geometry... finite number of points and lines
- Euclidean geometry: infinite number of points and lines. Construction: $\mathbb{R}^2$
- Finite geometries are often constructed using vector spaces over finite fields
- Recall: there exists a finite field of order q where q is a prime power

Applications

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- Finite geometry: coding theory
- Finite geometry: experimental designs

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- (P, L, I) - points, lines, incidence relation $I \subseteq P \times L$

Example: Generalised Polygons

Generalised n-gon

- Two points lie in at most one line, two lines intersect in at most one point
- No k-gons for $k \in \{3, \ldots, n-1\}$
- Any two elements (points or lines) is contained in an n-gon

- k-gon: sequence of points $a_0, \ldots, a_{k-1}$ where a_i and a_{i+1} lie in a common line ($+$ is mod k)

Example of Example: Generalised Triangle

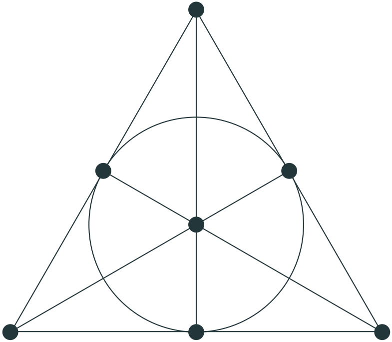

Figure 2: Fano Plane

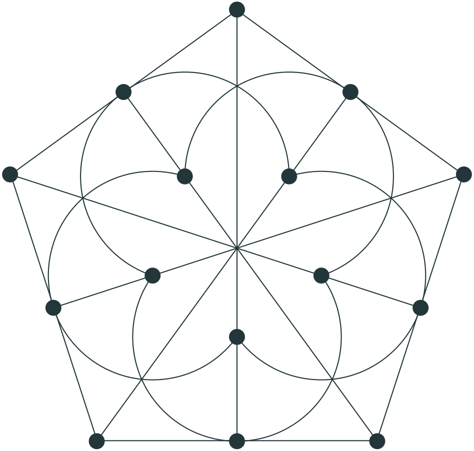

Figure 3: Cremona-Richmond Generalised Quadrangle

Alternative Definition of a Generalised Quadrangle

Generalised Quadrangle

- Two points lie in at most one line, two lines intersect in at most one point
- Given a line L and a point x not on L, there is a unique point y on L such that x and y are on a line

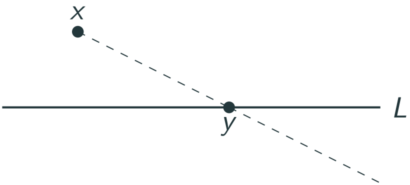

Figure 4: Second GQ Axiom

Graph Theory: Incidence Graph

- Vertices: Points and lines
- Edges: Two vertices are adjacent if they are incident
- Bipartite graph with diameter n and girth $2n$
- Diameter: Greatest distance between pairs of vertices
- Girth: Length of the shortest cycle in the graph
- Incident point-line pair is called a flag

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- Order (s, t)
- Feit and Higman proved: finite generalised n-gon of order (s, t), $s, t > 2$ implies that $n \in \{2, 3, 4, 6, 8\}$

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- So can we classify them?
- Symmetry conditions on its group of automorphisms: point-primitivity, line-transitivity etc.

Results on Generalised Quadrangles

- Bamberg, Giudici, Morris, Royle, Spiga (BGMRS) - 2011
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- Then G is almost simple
- Almost simple: $S \leqslant G \leqslant \mathrm{Aut}(S)$, S non-abelian and simple

- Bamberg, Evans - 2021
- No sporadic almost simple group can act primitively on points of any generalised quadrangle

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- If G is also flag-transitive then G is almost simple of Lie type
- Take G to be $\mathrm{Sz}(q)$ and $\mathrm{Ree}(q)$ where q is a prime power

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- Consider maximal subgroups of $\mathrm{Sz}(q)$ and $\mathrm{Ree}(q)$

Maximal Subgroups of Suzuki Groups

- $q = 2^m$, m odd
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- $\mathrm{Sz}(q_0)$ where $q = q_0^r$ with r prime and $q_0 > 2$

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- $\mathrm{Ree}(q_0)$ where $q = q_0^r$ with r prime