Block-colourings of e-star systems

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Joint work with Iren Darijani

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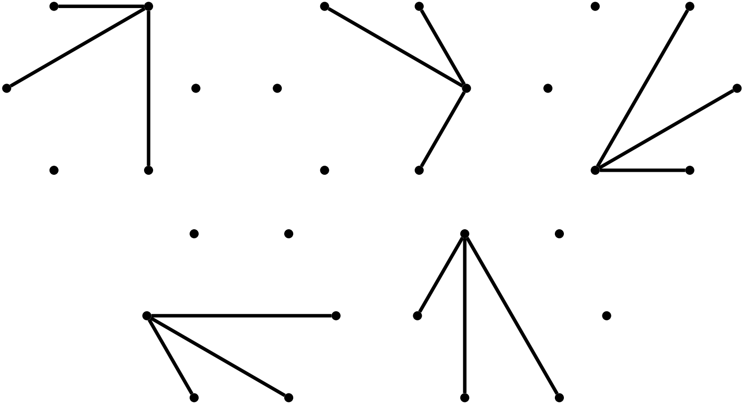
- **Definition:** An *e-star system* is a $K_{1,e}$-design.
- Since a 1-star is the same as K_2 (boring....), and a 2-star is the same as a path P_3, we will assume that $e \geq 3$.

An example

The following is a 3-star system of order 6:



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- Some examples include 1-factorizations ($G = K_2$), Kirkman triple systems ($G = K_3 = C_3$), and the uniform Oberwolfach problem ($G = C_m$).

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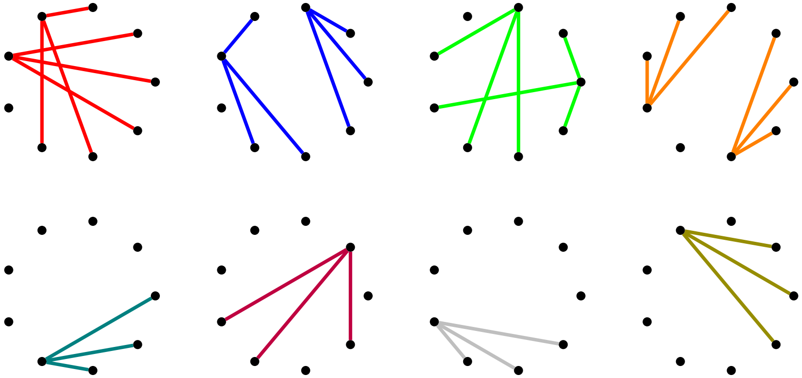
- If $\mathcal{D}$ is resolvable, then the chromatic index is as small as possible. So the interesting question is this: what is the least possible chromatic index of a G-design when no resolvable example can exist?

Colouring blocks: an example

An 8-block-colouring of a 3-star system of order 9:



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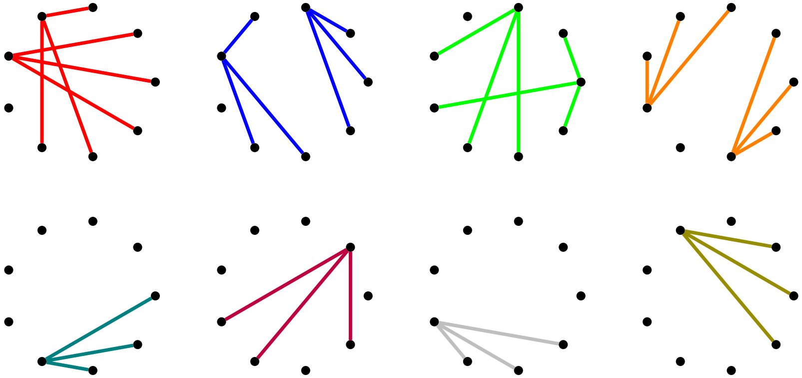

It turns out (by computer search) that 8 colours is the best possible for such a system.

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- What about an upper bound?

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- **Theorem:** (B+Darijani, 2023) For every admissible order n, there exists a 3-star system of order n with chromatic index at most n.
- Asymptotically, these are best-possible: for fixed e, there is a lower bound of $\Omega(n)$ and an upper bound of $O(n)$ on the minimum chromatic index.

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- Suppose that $F_1 = \{(V_1, V_2),\ (V_3, V_4),\ \ldots,\ (V_{2t-1}, V_{2t})\}$. The edges between V_{2j-1} and V_{2j} form a complete bipartite graph $K_{2e,2e}$; these can be decomposed into $2e$ colour classes of e-stars.

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- We can decompose the edges of $F_2, \ldots, F_{2t-1}$ in a similar way.

Idea of proof, II

- Altogether, we use $2e$ colours on each of the $2t-1$ 1-factors, and a further $2e-1$ colours within each $\mathcal{B}_i$, for a total of $2e(2t-1)+2e-1=4et-1=n-1$ colours.

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- For the other cases, the modifications needed sometimes require an additional colour, and the construction yields an n-block-colourable system.

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- So the *actual* values of the minimum chromatic index are still unknown.....

Hvala!

Reference: R. F. Bailey and I. Darijani, Block colourings of star systems, *Discrete Math.* **346** (2023), 113404 (14pp).