A Framework for Classifying Cocyclic HMs of Order $8p$

Santiago Barrera Acevedo

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Hadamard matrices

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where I_n is the identity of order n.

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- Hadamard showed that the order of a HM is necessarily 1, 2 or $4n$ for $n \in \mathbb{N}$.

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Equivalence of Hadamard matrices

- A driving force behind HM research is the **Hadamard Conjecture**, which asserts that for every positive integer n there exists a HM of order $4n$.[‡]

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- Since the number of HMs of order $4n$ appears to grow rapidly with n (which contrasts with the Hadamard Conjecture), it is necessary to introduce an equivalence relation on the set of HMs.

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- The group $\mathbf{Mon}(\boldsymbol{n}, \{\pm 1\})$ of all pairs of $\{\pm 1\}$-monomial matrices (signed permutation matrices) of order n acts on the set of $\{\pm 1\}$-matrices of order n via

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- Two HMs H and H' are **equivalent** if they lie in the same $\mathbf{Mon}(\boldsymbol{n}, \{\pm 1\})$-orbit.

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Classification of Hadamard matrices

- The classification of HMs of orders less than 30, up to equivalence, was achieved through the efforts of numerous mathematicians in the 1980s and 1990s[†]

n	1	2	3	4	5	6	7
# classes	1	1	1	5	3	60	487

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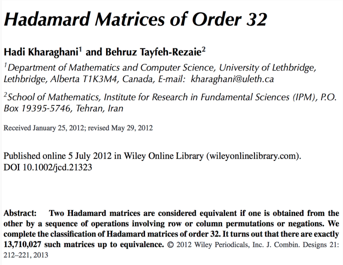

Hadamard Matrices of Order 32

Hadi Kharaghani[1] **and Behruz Tayfeh-Rezaie**[2]

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[2]*School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*

Received January 25, 2012; revised May 29, 2012

Published online 5 July 2012 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jcd.21323

Abstract: Two Hadamard matrices are considered equivalent if one is obtained from the other by a sequence of operations involving row or column permutations or negations. We complete the classification of Hadamard matrices of order 32. It turns out that there are exactly 13,710,027 such matrices up to equivalence. © 2012 Wiley Periodicals, Inc. J. Combin. Designs 21: 212–221, 2013

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- The classification of HMs of order 32, up to equivalence, was achieved in 2012.‡.
- There are exactly **13, 710, 027** equivalence classes of HMs.
- Given the profusion of equivalence classes of HMs, even at small orders, it makes sense to ask for classifications of HMs of special types.

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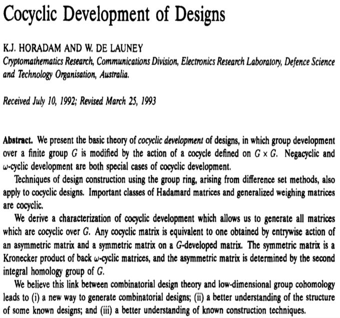

Cocyclic Development of Designs

K.J. HORADAM AND W. DE LAUNEY
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Abstract. We present the basic theory of *cocyclic development* of designs, in which group development over a finite group G is modified by the action of a cocycle defined on $G \times G$. Negacyclic and ω-cyclic development are both special cases of cocyclic development.

Techniques of design construction using the group ring, arising from difference set methods, also apply to cocyclic designs. Important classes of Hadamard matrices and generalized weighing matrices are cocyclic.

We derive a characterization of cocyclic development which allows us to generate all matrices which are cocyclic over G. Any cocyclic matrix is equivalent to one obtained by entrywise action of an asymmetric matrix and a symmetric matrix on a G-developed matrix. The symmetric matrix is a Kronecker product of back ω-cyclic matrices, and the asymmetric matrix is determined by the second integral homology group of G.

We believe this link between combinatorial design theory and low-dimensional group cohomology leads to (i) a new way to generate combinatorial designs; (ii) a better understanding of the structure of some known designs; and (iii) a better understanding of known construction techniques.

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We believe this link between combinatorial design theory and low-dimensional group cohomology leads to (i) a new way to generate combinatorial designs; (ii) a better understanding of the structure of some known designs; and (iii) a better understanding of known construction techniques.

- Let G be a finite group and let A be a $\mathbb{Z}G$-module. A **2-cocycle**‡ (or simply cocycle) with coefficients in A is a map

$$\psi : G \times G \to A \quad \text{such that}$$

$$\psi(g,h)\psi(gh,k) = \psi(h,k)^g \psi(g,hk), \text{ for all } g,h,k \in G. \qquad (3)$$

‡Up to equivalence of extensions, central extensions of A by G can be parameterised by the group $Z(G,A) = \{\psi : G \times G \to A \mid \psi(g,h)\psi(gh,k) = \psi(h,k)\psi(g,hk), \text{ for all } g,h,k \in G\}$

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- A **coboundary** is a cocycle of the form $\psi(g,h) = \phi(g)\phi(h)\phi(gh)^{-1}$ for a map $\phi : G \to A$.

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- In the following, let $A = C_2 = \langle -1 \rangle$ (with trivial $\mathbb{Z}G$-action).

  A HM H of order $4n$ is **cocyclic** with **indexing group** $G = \{g_1, \ldots, g_{4n}\}$ if there exist a 2-cocycle $\psi : G \times G \to \langle -1 \rangle$ and a map $\phi \colon G \to \langle -1 \rangle$ such that

$$H \equiv \left[\psi(g_i, g_j)\phi(g_i g_j)\right]_{i,j} \tag{4}$$

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- Group developed HMs are known to have square order.[†]
- Note $H \equiv [\psi(g_i, g_j)\phi(gh)\phi(g)\phi(h)]_{i,j}$; we work with this matrix instead.

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Classification of Cocyclic Hadamard matrices

- In 2010, Ó Cathaín and Röder reported the classification of CHMs of order less than 40.

The cocyclic Hadamard matrices of order less than 40

Padraig Ó Catháin · Marc Röder

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Abstract In this paper all cocyclic Hadamard matrices of order less than 40 are classified. That is, all such Hadamard matrices are explicitly constructed, up to Hadamard equivalence. This represents a significant extension and completion of work by de Launey and Ito. The theory of cocyclic development is discussed, and an algorithm for determining whether a given Hadamard matrix is cocyclic is described. Since all Hadamard matrices of order at most 28 have been classified, this algorithm suffices to classify cocyclic Hadamard matrices of order at most 28. Not even the total numbers of Hadamard matrices of orders 32 and 36 are known. Thus we use a different method to construct all cocyclic Hadamard matrices at these orders. A result of de Launey, Flannery and Horadam on the relationship between cocyclic Hadamard matrices and relative difference sets is used in the classification of cocyclic Hadamard matrices of orders 32 and 36. This is achieved through a complete enumeration and construction of $(4t, 2, 4t, 2t)$-relative difference sets in the groups of orders 64 and 72.

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- To achieve this, they used a known connection between CHMs and certain semi-regular $(4n, 2, 4n, 2n)$ relative difference sets in groups of order $8n$.[†]

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# classes	1	1	1	5	3	16	6	100	35

[†] de Launey, Flannery, and Horadam, Cocyclic Hadamard matrices and difference sets (2000)

Structure of cocyclic Hadamard matrices of order $4p$

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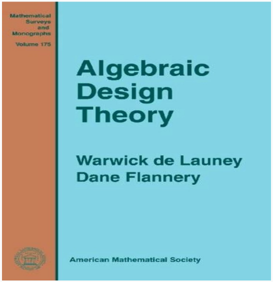

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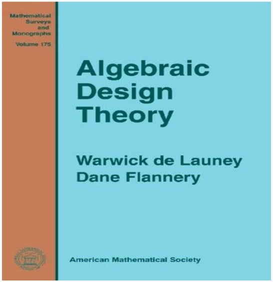
- de Launey and Flannery studied the structure of CHM of order $4p$.
- They showed that such matrices have indexing groups $K \ltimes C_p$, where $|K| = 4$, and can be described by a set of block arrays.



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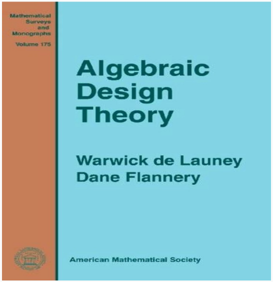
- Every CHM of order $4p$ and $p > 3$ prime with indexing group $K \ltimes C_p$ and cocycle ψ is equivalent to a matrix



$$\begin{bmatrix} W & X^a & Y^b & Z^{ab} \\ X & (-1)^r W^a & Z^b & (-1)^r Y^{ab} \\ Y & (-1)^t Z^a & (-1)^s W^b & (-1)^{s+t} X^{ab} \\ Z & (-1)^{r+t} Y^a & (-1)^s X^b & (-1)^{r+s+t} W^{ab} \end{bmatrix}$$

where $(r, s, t) \in \{(1,0,0), (0,1,0), (1,1,0), (1,1,1)\}$ depends on ψ, the blocks W, X, Y, Z are back-circulant, and a block M^x is circulant if and only if $x \in \{a, b, ab\} \subseteq K$ acts by inversion on C_p.[†]

[†] De Launey and Flannery: Algebraic Design Theory (2011)

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Constructing cocyclic Hadamard matrices of order 4*p*

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[1]School of Mathematics, Monash University, Clayton, Victoria, Australia

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- In 2019, Barrera Acevedo, Ó Cathaín and Dietrich recovered the aforementioned 4×4 block arrays via a group theoretical approach.
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p	3	5	7	11	13
# classes	1	1	3	63	336

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- They are currently exploring the idea of using SAT-solvers to classify CHMs of orders $4 \cdot 17$ and $4 \cdot 19$.

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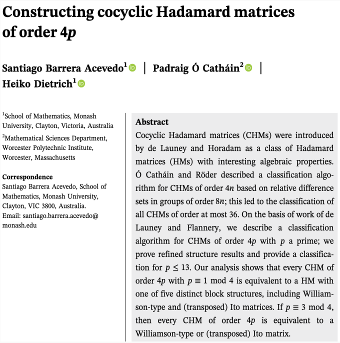

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Abstract
Cocyclic Hadamard matrices (CHMs) were introduced by de Launey and Horadam as a class of Hadamard matrices (HMs) with interesting algebraic properties. Ó Catháin and Röder described a classification algorithm for CHMs of order $4n$ based on relative difference sets in groups of order $8n$; this led to the classification of all CHMs of order at most 36. On the basis of work of de Launey and Flannery, we describe a classification algorithm for CHMs of order $4p$ with p a prime; we prove refined structure results and provide a classification for $p \leq 13$. Our analysis shows that every CHM of order $4p$ with $p \equiv 1 \bmod 4$ is equivalent to a HM with one of five distinct block structures, including Williamson-type and (transposed) Ito matrices. If $p \equiv 3 \bmod 4$, then every CHM of order $4p$ is equivalent to a Williamson-type or (transposed) Ito matrix.

p	3	5	7	11	13
# classes	1	1	3	63	336

It is natural to ask whether CHMs of orders $8p$ and $4pq$, for $2 < p < q$ primes, can be described by a set of block arrays, as in the case $4p$.

Cocyclic Hadamard matrices of order $8p$

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- In the following, let H be a CHM of order $8p$ with $p > 3$ prime, indexing group G and cocycle ψ.

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- $G \cong K \ltimes N$, where $|K| = 8$ and $N \cong C_p$, except for $G = C_7 \ltimes C_2^3$ – However there are no CHMs with indexing group $C_7 \ltimes C_2^3$ as $H^2(C_7 \ltimes C_2^3, C_2)$ is trivial.

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- $E_\psi = G \ltimes |_\psi \langle -1 \rangle = \hat{K} \ltimes \hat{N}$, where $\hat{K} \cong K \ltimes_\psi \langle -1 \rangle$ (here ψ denotes the restriction of $\psi : G \times G \to \langle -1 \rangle$ to $K \times K$) and $\hat{N} \cong C_p$.

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- If $K = C_8$ then $\hat{K} = C_{16}$ or $C_8 \times C_2$ (both which are disqualified due to Ito's and the fact that H is not group developed).

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- If $K = C_8$ then $\hat{K} = C_{16}$ or $C_8 \times C_2$ (both which are disqualified due to Ito's and the fact that H is not group developed).
- $K \in \{C_2^3, C_4 \times C_2, D_8, Q_8\}$ (all polycyclic groups).

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Combining ideas of Ó Cathaín and Röder[†], and Barrera Acevedo et al[‡]. we have the following result.

Theorem

Let H be a CHM of order $8p$ with indexing group $G = K \ltimes N$ and cocycle ψ. Then

$$H \equiv \left[\psi(k_i, k_j)\left[\phi\left(k_i k_j n^{k_j} m\right)\right]_{n,m \in N}\right]_{k_i, k_j \in K} \tag{5}$$

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For fixed k_i, k_j each inner $p \times p$ block $\left[\phi \left(k_i k_j n^{k_j} m \right) \right]_{n,m \in N}$ is group developed over N with respect to the action of K on N.

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Every matrix of form (5) is also cocyclic.

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K	Isomorphism type of $\hat{K}$	GAP ID [16, #]
$C_4 \times C_2$	$C_2 \ltimes (C_4 \times C_2)$, C_4^2	[16, 3], [16, 4]
	$C_2 \times C_8$, $C_2 \ltimes C_8$	[16, 5], [16, 6]
	$C_2^2 \times C_4$	[16, 10]
C_2^3	$C_2 \ltimes (C_4 \times C_2)$, $D_8 \times C_2$	[16, 3], [16, 11]
	C_2^4, $Q_8 \times C_2$	[16, 14], [16, 12]
D_8	$C_2 \ltimes (C_4 \times C_2)$, C_4^2	[16, 3], [16, 4]
	D_{16}, SD_{16}	[16, 7], [16, 8]
	Q_{16}, $D_8 \times C_2$	[16, 9], [16, 11]
Q_8	C_4^2, $Q_8 \times C_2$	[16, 4], [16, 12]

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	$C_2 \times C_8$, $C_2 \ltimes C_8$	$[16,5]$, $[16,6]$
	$C_2^2 \times C_4$	$[16,10]$
C_2^3	$C_2 \ltimes (C_4 \times C_2)$, $D_8 \times C_2$	$[16,3]$, $[16,11]$
	C_2^4, $Q_8 \times C_2$	$[16,14]$, $[16,12]$
D_8	$C_2 \ltimes (C_4 \times C_2)$, C_4^2	$[16,3]$, $[16,4]$
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For each isomorphism type of $\hat{K}$ we compute a representative cocycle.

Cocycles

Let $K = C_4 \times C_2$ and consider the presentation

$$K = \langle a, b, c \mid a^2 = 1, b^2 = c, c^2 = 1, b^a = b, c^a = c, c^b = c \rangle.$$

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The possible central extensions of K by C_2 are given by

$$\hat{K} = L_{i,k,r} = \langle a, b, c, z \mid a^2 = z^i, b^2 = c, c^2 = z^k, b^a = bz^r, z^2 = 1 \rangle$$

with $(i, k, r) \in \mathbb{Z}_2^3$.

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with $(i, k, r) \in \mathbb{Z}_2^3$.

From the central extension $1 \to C_2 \xrightarrow{\iota} L_{i,k,r} \xrightarrow{\pi} K \to 1$ take a lift $l : K \to L_{i,k,r}$ and compute the 2–cocycle

$$\psi_{i,k,r}(u, v) = \iota^{-1}(l(u)l(v)l(uv)^{-1}). \tag{6}$$

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$$[\psi_{i,k,r}(u, v)]_{u,v \in K} =$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & (-1)^i & 1 & 1 & (-1)^i & (-1)^i & 1 & (-1)^i \\ 1 & (-1)^r & 1 & 1 & (-1)^r & (-1)^r & (-1)^k & (-1)^{k+r} \\ 1 & 1 & 1 & (-1)^k & 1 & (-1)^k & (-1)^k & (-1)^k \\ 1 & (-1)^{i+r} & 1 & 1 & (-1)^{i+r} & (-1)^{i+r} & (-1)^k & (-1)^{i+k+r} \\ 1 & (-1)^i & 1 & (-1)^k & (-1)^i & (-1)^{i+k} & (-1)^k & (-1)^{i+k} \\ 1 & (-1)^r & (-1)^k & (-1)^k & (-1)^{k+r} & (-1)^{k+r} & (-1)^k & (-1)^{k+r} \\ 1 & (-1)^{i+r} & (-1)^k & (-1)^k & (-1)^{i+k+r} & (-1)^{i+k+r} & (-1)^k & (-1)^{i+k+r} \end{bmatrix}.$$

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The elements in the group of couboudaries $B^2(C_4 \times C_2, C_2)$ are determined as follows:

$$C_{\alpha,\beta,\gamma,\delta,\varepsilon} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \gamma & \delta & \gamma & \delta & \beta\gamma\varepsilon & \gamma\beta\varepsilon \\ 1 & \gamma & \alpha & \beta & \alpha\gamma\delta & \gamma\delta\varepsilon & \alpha\beta & \alpha\gamma\varepsilon \\ 1 & \delta & \beta & 1 & \varepsilon & \delta & \beta & \varepsilon \\ 1 & \gamma & \alpha\gamma\delta & \varepsilon & \alpha & \beta\gamma\delta & \alpha\beta\gamma & \alpha\varepsilon \\ 1 & \delta & \gamma\delta\varepsilon & \delta & \beta\gamma\delta & 1 & \beta\gamma\delta & \gamma\delta\varepsilon \\ 1 & \beta\gamma\varepsilon & \alpha\beta & \beta & \alpha\beta\gamma & \beta\gamma\delta & \alpha & \alpha\beta\gamma\delta\varepsilon \\ 1 & \beta\gamma\varepsilon & \alpha\gamma\varepsilon & \varepsilon & \alpha\varepsilon & \gamma\delta\varepsilon & \alpha\beta\gamma\delta\varepsilon & \alpha \end{bmatrix},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon \in \langle -1 \rangle$.

Block structure - Example

From the description

$$H \equiv \left[\psi(k_i, k_j)\left[\phi\left(k_i k_j n^{k_j} m\right)\right]_{n,m\in N}\right]_{k_i,k_j\in K}.$$

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Every CHM with indexing group $G \equiv (C_4 \times C_2) \ltimes C_p$ is equivalent to a matrix

$$\mathcal{H}_1(S,T,U,V,W,X,Y,Z)^{a,b,c,i,k,r}_{\alpha,\beta,\gamma,\delta,\varepsilon} =$$

$$\begin{bmatrix}
S & T^{a} & U^{b} & V^{c} & W^{ab} & X^{ac} & Y^{bc} & Z^{abc} \\
T & (-1)^{i} S^{a} & W^{b} & X^{c} & (-1)^{i} U^{ab} & (-1)^{i} V^{ac} & Z^{bc} & (-1)^{i} Y^{abc} \\
U & (-1)^{r} W^{a} & V^{b} & Y^{c} & (-1)^{r} X^{ab} & (-1)^{r} Z^{ac} & (-1)^{k} S^{bc} & (-1)^{k+r} T^{abc} \\
V & X^{a} & Y^{b} & (-1)^{k} S^{c} & Z^{ab} & (-1)^{k} T^{ac} & (-1)^{k} U^{bc} & (-1)^{k} W^{abc} \\
W & (-1)^{i+r} U^{a} & X^{b} & Z^{c} & (-1)^{i+r} V^{ab} & (-1)^{i+r} Y^{ac} & (-1)^{k} T^{bc} & (-1)^{i+k+r} S^{abc} \\
X & (-1)^{i} V^{a} & Z^{b} & (-1)^{k} T^{b} & (-1)^{i} Y^{ab} & (-1)^{i+k} S^{ac} & (-1)^{k} W^{bc} & (-1)^{i+k} U^{abc} \\
Y & (-1)^{r} Z^{a} a & (-1)^{k} S^{b} & (-1)^{k} U^{b} & (-1)^{k+r} T^{ab} & (-1)^{k+r} W^{ac} & (-1)^{k} V^{bc} & (-1)^{k+r} X^{abc} \\
Z & (-1)^{i+r} Y^{a} & (-1)^{k} T^{b} & (-1)^{k} W^{b} & (-1)^{i+k+r} S^{ab} & (-1)^{i+k+r} U^{ac} & (-1)^{k} X^{bc} & (-1)^{i+k+r} V^{abc}
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$$(C_{\alpha,\beta,\gamma,\delta,\varepsilon} \otimes J_p),$$

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V & X^a & Y^b & (-1)^k S^c & Z^{ab} & (-1)^k T^{ac} & (-1)^k U^{bc} & (-1)^k W^{abc} \\
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\end{bmatrix} \odot$$

$$(C_{\alpha,\beta,\gamma,\delta,\varepsilon} \otimes J_p),$$

where $C_4 \times C_2 = \langle a, b, c\rangle$, $i, k, r \in \{0, 1\}$, $\alpha, \beta, \gamma, \delta, \varepsilon \in \langle -1\rangle$, J_p denotes the all 1's matrix of size $p \times p$, and $\otimes$ and $\odot$ denote the Kronecker and Hadamard products of matrices, respectively.

Block structure

Theorem

Every CHM H of order $8p$, with $p > 3$ prime, and indexing group $G = K \ltimes N$, where $|K| = 8$ and $N \cong C_p$, is equivalent to one of four block matrices:

$$H_1 = \mathcal{H}_1(S, T, U, V, W, X, Y, Z)^{a,b,c,i,k,r}_{\alpha,\beta,\gamma,\delta,\varepsilon} \quad \text{for } K = C_4 \times C_2$$

$$H_2 = \mathcal{H}_2(\mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}, \mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})^{a,b,c,i,j,k,r,s,t}_{\alpha,\beta,\gamma,\delta} \quad \text{for } K = C_2^3$$

$$H_3 = \mathcal{H}_3(S, T, U, V, W, X, Y, Z)^{a,b,c,i,j,k}_{\alpha,\beta,\gamma,\delta,\varepsilon} \quad \text{for } K = D_8$$

$$H_4 = \mathcal{H}_4(S, T, U, V, W, X, Y, Z)^{a,b,c,i,r}_{\alpha,\beta,\gamma,\delta,\varepsilon} \quad \text{for } K = Q_8$$

where $(C_4 \times C_2) = \langle a, b, c \rangle$, $i, k, r \in \{0, 1\}$ *and* $\alpha, \beta, \gamma, \delta, \varepsilon \in \langle -1 \rangle$.

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Ideas to trim the search space:

- Reducing the coboundary space.
- Establishing Hadamard equivalences that preserve the block structures.
- Controlling eigenvalues of the block matrices.

Coboundary space reduction

Can we get rid of coboundaries?

Hola

Coboundary space reduction

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For example, let $G \equiv (C_2^3) \ltimes C_p$.

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S & T^a & U^b & V^c & W^{ab} & X^{ac} & Y^{bc} & Z^{abc} \\
T & (-1)^i S^a & \alpha W^b & \beta X^c & (-1)^i \alpha U^{ab} & (-1)^i \beta V^{ac} & \delta Z^{bc} & (-1)^i \delta Y^{abc} \\
U & (-1)^r \alpha W^a & (-1)^j S^b & \gamma Y^c & (-1)^{j+r} \alpha T^{ab} & (-1)^r \alpha\beta\delta Z^{ac} & (-1)^j \gamma j V^{bc} & (-1)^{j+r} \beta\gamma\delta X^{abc} \\
V & (-1)^s \beta X^a & (-1)^t \gamma Y^b & (-1)^k S^c & (-1)^{s+t} \alpha\gamma\delta Z^{ab} & (-1)^{k+s} \beta T^{ac} & (-1)^{k+t} \gamma U^{bc} & (-1)^{k+s+t} \alpha\gamma\delta W^{abc} \\
W & (-1)^{i+r} \alpha U^a & (-1)^j \alpha T^b & \alpha\gamma\delta Z^c & (-1)^{i+j+r} S^{ab} & (-1)^{i+r} \alpha\beta\gamma Y^{ac} & (-1)^j \alpha\beta\gamma X^{bc} & (-1)^{i+j+r} \alpha\gamma\delta V^{abc} \\
X & (-1)^{i+s} \beta V^a & (-1)^t \beta\gamma\delta Z^b & (-1)^k \beta T^c & (-1)^{i+s+t} \alpha\beta\gamma Y^{ab} & (-1)^{i+k+s} S^{ac} & (-1)^{k+t} \alpha\beta\gamma W^{bc} & (-1)^{m-j-r} \beta\gamma\delta U^{abc} \\
Y & (-1)^{r+s} \delta Z^a & (-1)^{j+t} \gamma V^b & (-1)^k \gamma U^c & (-1)^{m-i-k} \alpha\beta\gamma X^{ab} & (-1)^{k+r+s} \alpha\beta\gamma W^{ac} & (-1)^{j+k+t} S^{bc} & (-1)^{m-i} \delta T^{abc} \\
Z & (-1)^{i+r+s} \delta Y^a & (-1)^{j+t} \beta\gamma\delta X^b & (-1)^k \alpha\gamma\delta W^c & (-1)^{m-k} \alpha\gamma\delta V^{ab} & (-1)^{m-j-t} \beta\gamma\delta U^{ac} & (-1)^{j+k+t} \delta T^{bc} & (-1)^m S^{abc}
\end{bmatrix}$$

Hola

Coboundary space reduction

Can we get rid of coboundaries?

For example, let $G \equiv (C_2^3) \ltimes C_p$.

$$\mathcal{H}_2(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k,r,s,t}_{\alpha,\beta,\gamma,\delta} \equiv$$

$$\begin{bmatrix}
S & T^a & U^b & \gamma\mathbf{V}^{\mathbf{c}} & \alpha\mathbf{W}^{\mathbf{ab}} & \beta\gamma\mathbf{X}^{\mathbf{ac}} & Y^{bc} & \delta\mathbf{Z}^{\mathbf{abc}} \\
T & (\text{-}1)^i S^a & \alpha\mathbf{W}^{\mathbf{b}} & \beta\gamma\mathbf{X}^{\mathbf{c}} & (\text{-}1)^i U^{ab} & (-1)^i \gamma\mathbf{V}^{\mathbf{ac}} & \delta\mathbf{Z}^{\mathbf{bc}} & (\text{-}1)^i Y^{abc} \\
U & (\text{-}1)^r \alpha\mathbf{W}^{\mathbf{a}} & (\text{-}1)^j S^b & Y^c & (\text{-}1)^{j+r} T^{ab} & (\text{-}1)^r \delta\mathbf{Z}^{\mathbf{ac}} & (\text{-}1)^j \gamma\mathbf{V}^{\mathbf{bc}} & (\text{-}1)^{j+r} \beta\gamma\mathbf{X}^{\mathbf{abc}} \\
\gamma V & (\text{-}1)^s \beta\gamma\mathbf{X}^{\mathbf{a}} & (\text{-}1)^t Y^b & (\text{-}1)^k S^c & (\text{-}1)^{s+t} \delta\mathbf{Z}^{\mathbf{ab}} & (\text{-}1)^{k+s} T^{ac} & (\text{-}1)^{k+t} U^{bc} & (\text{-}1)^{k+s+t} \alpha\mathbf{W}^{\mathbf{abc}} \\
\alpha\mathbf{W} & (\text{-}1)^{i+r} U^a & (\text{-}1)^j T^b & \delta\mathbf{Z}^{\mathbf{c}} & (\text{-}1)^{i+j+r} S^{ab} & (\text{-}1)^{i+r} Y^{ac} & (\text{-}1)^j \beta\gamma\mathbf{X}^{\mathbf{bc}} & (\text{-}1)^{i+j+r} \gamma\mathbf{V}^{\mathbf{abc}} \\
\beta\gamma\mathbf{X} & (\text{-}1)^{i+s} \gamma\mathbf{V}^{\mathbf{a}} & (\text{-}1)^t \delta\mathbf{Z}^{\mathbf{b}} & (\text{-}1)^k T^c & (\text{-}1)^{i+s+t} Y^{ab} & (\text{-}1)^{i+k+s} S^{ac} & (\text{-}1)^{k+t} \alpha\mathbf{W}^{\mathbf{bc}} & (\text{-}1)^{m\text{-}j\text{-}r} U^{abc} \\
Y & (\text{-}1)^{r+s} \delta\mathbf{Z}^{\mathbf{a}} & (\text{-}1)^{j+t} \gamma\mathbf{V}^{\mathbf{b}} & (\text{-}1)^k U^c & (\text{-}1)^{m\text{-}i\text{-}k} \beta\gamma\mathbf{X}^{\mathbf{ab}} & (\text{-}1)^{k+r+s} \alpha\mathbf{W}^{\mathbf{ac}} & (\text{-}1)^{j+k+t} S^{bc} & (\text{-}1)^{m\text{-}i} T^{abc} \\
\delta\mathbf{Z} & (\text{-}1)^{i+r+s} Y^a & (\text{-}1)^{j+t} \beta\gamma\mathbf{X}^{\mathbf{b}} & (\text{-}1)^k \alpha\mathbf{W}^{\mathbf{c}} & (\text{-}1)^{m\text{-}k} \gamma\mathbf{V}^{\mathbf{ab}} & (\text{-}1)^{m\text{-}j\text{-}t} U^{ac} & (\text{-}1)^{j+k+t} T^{bc} & (\text{-}1)^m S^{abc}
\end{bmatrix}$$

Coboundary space reduction

Can we get rid of coboundaries?

For example, let $G \equiv (C_2^3) \ltimes C_p$.

$$\mathcal{H}_2(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k,r,s,t}_{\alpha,\beta,\gamma,\delta} \equiv$$

$$\begin{bmatrix}
S & T^a & U^b & \gamma\mathbf{V^c} & \alpha\mathbf{W^{ab}} & \beta\gamma\mathbf{X^{ac}} & Y^{bc} & \delta\mathbf{Z^{abc}} \\
T & (\text{-}1)^i S^a & \alpha\mathbf{W^b} & \beta\gamma\mathbf{X^c} & (\text{-}1)^i U^{ab} & (-1)^i\gamma\mathbf{V^{ac}} & \delta\mathbf{Z^{bc}} & (\text{-}1)^i Y^{abc} \\
U & (\text{-}1)^r\alpha\mathbf{W^a} & (\text{-}1)^j S^b & Y^c & (\text{-}1)^{j+r} T^{ab} & (\text{-}1)^r\delta\mathbf{Z^{ac}} & (\text{-}1)^j\gamma\mathbf{V^{bc}} & (\text{-}1)^{j+r}\beta\gamma\mathbf{X^{abc}} \\
\gamma V & (\text{-}1)^s\beta\gamma\mathbf{X^a} & (\text{-}1)^t Y^b & (\text{-}1)^k S^c & (\text{-}1)^{s+t}\delta\mathbf{Z^{ab}} & (\text{-}1)^{k+s} T^{ac} & (\text{-}1)^{k+t} U^{bc} & (\text{-}1)^{k+s+t}\alpha\mathbf{W^{abc}} \\
\alpha\mathbf{W} & (\text{-}1)^{i+r} U^a & (\text{-}1)^j T^b & \delta\mathbf{Z^c} & (\text{-}1)^{i+j+r} S^{ab} & (\text{-}1)^{i+r} Y^{ac} & (\text{-}1)^j\beta\gamma\mathbf{X^{bc}} & (\text{-}1)^{i+j+r}\gamma\mathbf{V^{abc}} \\
\beta\gamma\mathbf{X} & (\text{-}1)^{i+s}\gamma\mathbf{V^a} & (\text{-}1)^t\delta\mathbf{Z^b} & (\text{-}1)^k T^c & (\text{-}1)^{i+s+t} Y^{ab} & (\text{-}1)^{i+k+s} S^{ac} & (\text{-}1)^{k+t}\alpha\mathbf{W^{bc}} & (\text{-}1)^{m\text{-}j\text{-}r} U^{abc} \\
Y & (\text{-}1)^{r+s}\delta\mathbf{Z^a} & (\text{-}1)^{j+t}\gamma\mathbf{V^b} & (\text{-}1)^k U^c & (\text{-}1)^{m\text{-}i\text{-}k}\beta\gamma\mathbf{X^{ab}} & (\text{-}1)^{k+r+s}\alpha\mathbf{W^{ac}} & (\text{-}1)^{j+k+t} S^{bc} & (\text{-}1)^{m\text{-}i} T^{abc} \\
\delta\mathbf{Z} & (\text{-}1)^{i+r+s} Y^a & (\text{-}1)^{j+t}\beta\gamma\mathbf{X^b} & (\text{-}1)^k\alpha\mathbf{W^c} & (\text{-}1)^{m\text{-}k}\gamma\mathbf{V^{ab}} & (\text{-}1)^{m\text{-}j\text{-}t} U^{ac} & (\text{-}1)^{j+k+t} T^{bc} & (\text{-}1)^m S^{abc}
\end{bmatrix}$$

Hence,

$$H_2 = \mathcal{H}_2(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k,r,s,t}_{\alpha,\beta,\gamma,\delta} \equiv \mathcal{H}_2(S,T,U,\gamma\mathbf{V},\alpha\mathbf{W},\beta\gamma\mathbf{X},Y,\delta\mathbf{Z})^{a,b,c,i,j,k,r,s,t}_{\mathbf{1,1,1,1}}$$

Coboundary space reduction

Proposition

Let

$$H_1 = \mathcal{H}_1(S,T,U,V,W,X,Y,Z)^{a,b,c,i,k,r}_{\alpha,\beta,\gamma,\delta,\varepsilon} \qquad H_2 = \mathcal{H}_2(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k,r,s,t}_{\alpha,\beta,\gamma,\delta}$$

$$H_3 = \mathcal{H}_3(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k}_{\alpha,\beta,\gamma,\delta,\varepsilon} \qquad H_4 = \mathcal{H}_4(S,T,U,V,W,X,Y,Z)^{a,b,c,i,r}_{\alpha,\beta,\gamma,\delta,\varepsilon}.$$

Then

$$H_1 \equiv \mathcal{H}_1(S,T,U,\alpha V,\gamma W,\alpha\delta X,\alpha\beta Y,\alpha\gamma\varepsilon Z)^{a,b,c,i,k,r}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_2 \equiv \mathcal{H}_2(S,T,U,\gamma V,\alpha W,\beta\gamma X,Y,\delta Z)^{a,b,c,i,j,k,r,s,t}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_3 \equiv \mathcal{H}_3(S,T,\beta\varepsilon U,V,W,X,\varepsilon\gamma Y,\delta Z)^{a,b,c,i,j,k}_{\alpha,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_4 \equiv \mathcal{H}_4(S,T,U,\alpha V,W,\beta X,\gamma Y,\delta Z)^{a,b,c,i,r}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\varepsilon}.$$

Coboundary space reduction

Proposition

Let

$$H_1 = \mathcal{H}_1(S,T,U,V,W,X,Y,Z)^{a,b,c,i,k,r}_{\alpha,\beta,\gamma,\delta,\varepsilon} \qquad H_2 = \mathcal{H}_2(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k,r,s,t}_{\alpha,\beta,\gamma,\delta}$$

$$H_3 = \mathcal{H}_3(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k}_{\alpha,\beta,\gamma,\delta,\varepsilon} \qquad H_4 = \mathcal{H}_4(S,T,U,V,W,X,Y,Z)^{a,b,c,i,r}_{\alpha,\beta,\gamma,\delta,\varepsilon}.$$

Then

$$H_1 \equiv \mathcal{H}_1(S,T,U,\alpha V,\gamma W,\alpha\delta X,\alpha\beta Y,\alpha\gamma\varepsilon Z)^{a,b,c,i,k,r}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_2 \equiv \mathcal{H}_2(S,T,U,\gamma V,\alpha W,\beta\gamma X,Y,\delta Z)^{a,b,c,i,j,k,r,s,t}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_3 \equiv \mathcal{H}_3(S,T,\beta\varepsilon U,V,W,X,\varepsilon\gamma Y,\delta Z)^{a,b,c,i,j,k}_{\alpha,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_4 \equiv \mathcal{H}_4(S,T,U,\alpha V,W,\beta X,\gamma Y,\delta Z)^{a,b,c,i,r}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\varepsilon}.$$

In the following, let

$$H_1 = \mathcal{H}_1(S,T,U,V,W,X,Y,Z)^{a,b,c,i,k,r}_{1,1,1,1,1} \qquad H_2 = \mathcal{H}_2(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k,r,s,t}_{1,1,1,1}$$

$$H_3 = \mathcal{H}_3(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k}_{\alpha,1,1,1,1} \qquad H_4 = \mathcal{H}_4(S,T,U,V,W,X,Y,Z)^{a,b,c,i,r}_{1,1,1,1,\varepsilon}.$$

Cobounday space reduction

Proposition

Let

$$H_1 = \mathcal{H}_1(S,T,U,V,W,X,Y,Z)^{a,b,c,i,k,r}_{\alpha,\beta,\gamma,\delta,\varepsilon} \qquad H_2 = \mathcal{H}_2(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k,r,s,t}_{\alpha,\beta,\gamma,\delta}$$

$$H_3 = \mathcal{H}_3(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k}_{\alpha,\beta,\gamma,\delta,\varepsilon} \qquad H_4 = \mathcal{H}_4(S,T,U,V,W,X,Y,Z)^{a,b,c,i,r}_{\alpha,\beta,\gamma,\delta,\varepsilon}.$$

Then

$$H_1 \equiv \mathcal{H}_1(S,T,U,\alpha V,\gamma W,\alpha\delta X,\alpha\beta Y,\alpha\gamma\varepsilon Z)^{a,b,c,i,k,r}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_2 \equiv \mathcal{H}_2(S,T,U,\gamma V,\alpha W,\beta\gamma X,Y,\delta Z)^{a,b,c,i,j,k,r,s,t}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_3 \equiv \mathcal{H}_3(S,T,\beta\varepsilon U,V,W,X,\varepsilon\gamma Y,\delta Z)^{a,b,c,i,j,k}_{\alpha,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}}$$

$$H_4 \equiv \mathcal{H}_4(S,T,U,\alpha V,W,\beta X,\gamma Y,\delta Z)^{a,b,c,i,r}_{\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\varepsilon}.$$

In the following, let

$$H_1 = \mathcal{H}_1(S,T,U,V,W,X,Y,Z)^{a,b,c,i,k,r} \qquad H_2 = \mathcal{H}_2(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k,r,s,t}$$

$$H_3 = \mathcal{H}_3(S,T,U,V,W,X,Y,Z)^{a,b,c,i,j,k}_{\alpha} \qquad H_4 = \mathcal{H}_4(S,T,U,V,W,X,Y,Z)^{a,b,c,i,r}_{\varepsilon}.$$

Block-structure-preserving equivalences

Can we multiply rows/columns of $H_1, \ldots, H_4$ by -1 preserving their block structure?

Block-structure-preserving equivalences

Can we multiply rows/columns of $H_1, \ldots, H_4$ by -1 preserving their block structure?

$$H_1 = \mathcal{H}_1(S,T,U,V,W,X,Y,Z)^{a,b,c,i,k,r} =$$

$$\begin{bmatrix}
S & T^a & U^b & V^c & W^{ab} & X^{ac} & Y^{bc} & Z^{abc} \\
T & (-1)^i S^a & W^b & X^c & (-1)^i U^{ab} & (-1)^i V^{ac} & Z^{bc} & (-1)^i Y^{abc} \\
U & (-1)^r W^a & V^b & Y^c & (-1)^r X^{ab} & (-1)^r Z^{ac} & (-1)^k S^{bc} & (-1)^{k+r} T^{abc} \\
V & X^a & Y^b & (-1)^k S^c & Z^{ab} & (-1)^k T^{ac} & (-1)^k U^{bc} & (-1)^k W^{abc} \\
W & (-1)^{i+r} U^a & X^b & Z^c & (-1)^{i+r} V^{ab} & (-1)^{i+r} Y^{ac} & (-1)^k T^{bc} & (-1)^{i+k+r} S^{abc} \\
X & (-1)^i V^a & Z^b & (-1)^k T^b & (-1)^i Y^{ab} & (-1)^{i+k} S^{ac} & (-1)^k W^{bc} & (-1)^{i+k} U^{abc} \\
Y & (-1)^r Z^a a & (-1)^k S^b & (-1)^k U^b & (-1)^{k+r} T^{ab} & (-1)^{k+r} W^{ac} & (-1)^k V^{bc} & (-1)^{k+r} X^{abc} \\
Z & (-1)^{i+r} Y^a & (-1)^k T^b & (-1)^k W^b & (-1)^{i+k+r} S^{ab} & (-1)^{i+k+r} U^{ac} & (-1)^k X^{bc} & (-1)^{i+k+r} V^{abc}
\end{bmatrix}$$

Thus,

$$H_1 = \pm 1\ \mathcal{H}_1(-S, T, U, -V, -W, X, Y, -Z)^{a,b,c,i,k,r}.$$

Block-structure-preserving equivalences

Can we multiply rows/columns of $H_1, \ldots, H_4$ by -1 preserving their block structure?

$$H_1 = \mathcal{H}_1(S, T, U, V, W, X, Y, Z)^{a,b,c,i,k,r} \equiv$$

$$\begin{bmatrix}
-S & T^a & U^b & -V^c & -W^{ab} & X^{ac} & Y^{bc} & -Z^{abc} \\
-T & (-1)^i S^a & W^b & -X^c & -(-1)^i U^{ab} & (-1)^i V^{ac} & Z^{bc} & -(-1)^i Y^{abc} \\
-U & (-1)^r W^a & V^b & -Y^c & -(-1)^r X^{ab} & (-1)^r Z^{ac} & (-1)^k S^{bc} & -(-1)^{k+r} T^{abc} \\
-V & X^a & Y^b & -(-1)^k S^c & -Z^{ab} & (-1)^k T^{ac} & (-1)^k U^{bc} & -(-1)^k W^{abc} \\
-W & (-1)^{i+r} U^a & X^b & -Z^c & -(-1)^{i+r} V^{ab} & (-1)^{i+r} Y^{ac} & (-1)^k T^{bc} & -(-1)^{i+k+r} S^{abc} \\
-X & (-1)^i V^a & Z^b & -(-1)^k T^b & -(-1)^i Y^{ab} & (-1)^{i+k} S^{ac} & (-1)^k W^{bc} & -(-1)^{i+k} U^{abc} \\
-Y & (-1)^r Z^a a & (-1)^k S^b & -(-1)^k U^b & -(-1)^{k+r} T^{ab} & (-1)^{k+r} W^{ac} & (-1)^k V^{bc} & -(-1)^{k+r} X^{abc} \\
-Z & (-1)^{i+r} Y^a & (-1)^k T^b & -(-1)^k W^b & -(-1)^{i+k+r} S^{ab} & (-1)^{i+k+r} U^{ac} & (-1)^k X^{bc} & -(-1)^{i+k+r} V^{abc}
\end{bmatrix}$$

Thus,

$$H_1 \equiv \pm 1\ \mathcal{H}_1(-S, T, U, -V, -W, X, Y, -Z)^{a,b,c,i,k,r}_{1,1,1,1}.$$

Block-structure-preserving equivalences

Can we multiply rows/columns of $H_1, \ldots, H_4$ by -1 preserving their block structure?

$H_1 = \mathcal{H}_1(S, T, U, V, W, X, Y, Z)^{a,b,c,i,k,r} \equiv$

$$\begin{bmatrix} -S & T^a & U^b & -V^c & -W^{ab} & X^{ac} & Y^{bc} & -Z^{abc} \\ T & -(-1)^i S^a & -W^b & X^c & (-1)^i U^{ab} & -(-1)^i V^{ac} & -Z^{bc} & (-1)^i Y^{abc} \\ U & -(-1)^r W^a & -V^b & Y^c & (-1)^r X^{ab} & -(-1)^r -Z^{ac} & -(-1)^k S^{bc} & (-1)^{k+r} T^{abc} \\ -V & X^a & Y^b & -(-1)^k S^c & -Z^{ab} & (-1)^k T^{ac} & (-1)^k U^{bc} & -(-1)^k W^{abc} \\ -W & (-1)^{i+r} U^a & X^b & -Z^c & -(-1)^{i+r} V^{ab} & (-1)^{i+r} Y^{ac} & (-1)^k T^{bc} & -(-1)^{i+k+r} S^{abc} \\ X & -(-1)^i V^a & -Z^b & (-1)^k T^b & (-1)^i Y^{ab} & -(-1)^{i+k} S^{ac} & -(-1)^k W^{bc} & (-1)^{i+k} U^{abc} \\ Y & -(-1)^r Z^a a & -(-1)^k S^b & (-1)^k U^b & (-1)^{k+r} T^{ab} & -(-1)^{k+r} W^{ac} & -(-1)^k V^{bc} & (-1)^{k+r} X^{abc} \\ -Z & (-1)^{i+r} Y^a & (-1)^k T^b & -(-1)^k W^b & -(-1)^{i+k+r} S^{ab} & (-1)^{i+k+r} U^{ac} & (-1)^k X^{bc} & -(-1)^{i+k+r} V^{abc} \end{bmatrix}$$

Block-structure-preserving equivalences

Can we multiply rows/columns of $H_1, \ldots, H_4$ by -1 preserving their block structure?

$H_1 = \mathcal{H}_1(S, T, U, V, W, X, Y, Z)^{a,b,c,i,k,r} \equiv$

$$\begin{bmatrix}
-S & T^a & U^b & -V^c & -W^{ab} & X^{ac} & Y^{bc} & -Z^{abc} \\
T & -(-1)^i S^a & -W^b & X^c & (-1)^i U^{ab} & -(-1)^i V^{ac} & -Z^{bc} & (-1)^i Y^{abc} \\
U & -(-1)^r W^a & -V^b & Y^c & (-1)^r X^{ab} & -(-1)^r - Z^{ac} & -(-1)^k S^{bc} & (-1)^{k+r} T^{abc} \\
-V & X^a & Y^b & -(-1)^k S^c & -Z^{ab} & (-1)^k T^{ac} & (-1)^k U^{bc} & -(-1)^k W^{abc} \\
-W & (-1)^{i+r} U^a & X^b & -Z^c & -(-1)^{i+r} V^{ab} & (-1)^{i+r} Y^{ac} & (-1)^k T^{bc} & -(-1)^{i+k+r} S^{abc} \\
X & -(-1)^i V^a & -Z^b & (-1)^k T^b & (-1)^i Y^{ab} & -(-1)^{i+k} S^{ac} & -(-1)^k W^{bc} & (-1)^{i+k} U^{abc} \\
Y & -(-1)^r Z^a a & -(-1)^k S^b & (-1)^k U^b & (-1)^{k+r} T^{ab} & -(-1)^{k+r} W^{ac} & -(-1)^k V^{bc} & (-1)^{k+r} X^{abc} \\
-Z & (-1)^{i+r} Y^a & (-1)^k T^b & -(-1)^k W^b & -(-1)^{i+k+r} S^{ab} & (-1)^{i+k+r} U^{ac} & (-1)^k X^{bc} & -(-1)^{i+k+r} V^{abc}
\end{bmatrix}$$

Thus,

$$H_1 \equiv \mathcal{H}_1(-S, T, U, -V, -W, X, Y, -Z)^{a,b,c,i,k,r}$$

Block-structure-preserving equivalences

Proposition

$H_1 \equiv \pm 1\ \mathcal{H}_1(-S, T, U, -V, -W, X, Y, -Z)^{a,b,c,i,k,r}$

$H_3 \equiv \pm 1 \mathcal{H}_3(-S, T, -U, -V, W, X, -Y, Z)_{\alpha}^{a,b,c,i,j,k,r,s,t}$

$$\begin{aligned} H_4 &\equiv \pm 1\ \mathcal{H}_4(-T, -V, W, -X, Y, -S, Z, U)_{\varepsilon}^{a,b,c,i,r} \equiv \pm 1\ \mathcal{H}_4(-T, V, -W, -X, Y, S, -Z, U)_{\varepsilon}^{a,b,c,i,r} \\ &\equiv \pm 1\ \mathcal{H}_4(-T, V, W, -X, -Y, S, Z, -U)_{\varepsilon}^{a,b,c,i,r} \equiv \pm 1\ \mathcal{H}_4(T, -V, -W, X, Y, -S, -Z, U)_{\varepsilon}^{a,b,c,i,r} \\ &\equiv \pm 1\ \mathcal{H}_4(T, -V, W, X, -Y, -S, Z, -U)_{\varepsilon}^{a,b,c,i,r} \equiv \pm 1\ \mathcal{H}_4(T, V, -W, X, -Y, S, -Z, -U)_{\varepsilon}^{a,b,c,i,r} \end{aligned}$$

$H_2 \equiv \pm 1 \mathcal{H}_2(e_1 S, e_2 T, e_3 U, e_4 V, e_5 W, e_6 X, e_7 Y, e_8 Z)^{a,b,c,i,j,k,r,s,t}$

where $e_l \in \{\pm 1\}$, *for* $l = 1, \dots, 8$, *and exactly three of* e_1, e_2, e_3, e_4 *and one of* e_5, e_6, e_7, e_8 *are* -1.

Block-structure-preserving equivalences

Can we rearrange the blocks of $H_1, \ldots, H_4$ preserving their structure?

Thus

$$H_1 \equiv \mathcal{H}_1(T, S, W, X, U, V, Z, Y)^{a,b,c,i,k,r}$$

Block-structure-preserving equivalences

Can we rearrange the blocks of $H_1, \ldots, H_4$ preserving their structure?

$$H_1 = \mathcal{H}_1(S,T,U,V,W,X,Y,Z)^{a,b,c,i,k,r} =$$

$$\begin{bmatrix}
S & T^a & U^b & V^c & W^{ab} & X^{ac} & Y^{bc} & Z^{abc} \\
T & (-1)^i S^a & W^b & X^c & (-1)^i U^{ab} & (-1)^i V^{ac} & Z^{bc} & (-1)^i Y^{abc} \\
U & (-1)^r W^a & V^b & Y^c & (-1)^r X^{ab} & (-1)^r Z^{ac} & (-1)^k S^{bc} & (-1)^{k+r} T^{abc} \\
V & X^a & Y^b & (-1)^k S^c & Z^{ab} & (-1)^k T^{ac} & (-1)^k U^{bc} & (-1)^k W^{abc} \\
W & (-1)^{i+r} U^a & X^b & Z^c & (-1)^{i+r} V^{ab} & (-1)^{i+r} Y^{ac} & (-1)^k T^{bc} & (-1)^{i+k+r} S^{abc} \\
X & (-1)^i V^a & Z^b & (-1)^k T^b & (-1)^i Y^{ab} & (-1)^{i+k} S^{ac} & (-1)^k W^{bc} & (-1)^{i+k} U^{abc} \\
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Block-structure-preserving equivalences

Can we rearrange the blocks of $H_1, \ldots, H_4$ preserving their structure?

$$H_1 = \mathcal{H}_1(S, T, U, V, W, X, Y, Z)^{a,b,c,i,k,r} \equiv$$

$$\begin{bmatrix} T & S^a & W^b & X^c & U^{ab} & V^{ac} & Z^{bc} & Y^{abc} \\ S & (-1)^i T^a & U^b & V^c & (-1)^i W^{ab} & (-1)^i X^{ac} & Y^{bc} & (-1)^i Z^{abc} \\ W & (-1)^r U^a & X^b & Z^c & (-1)^r V^{ab} & (-1)^r Y^{ac} & (-1)^k T^{bc} & (-1)^{k+r} S^{abc} \\ X & V^a & Z^b & (-1)^k T^b & Y^{ab} & (-1)^k S^{ac} & (-1)^k W^{bc} & (-1)^k U^{abc} \\ U & (-1)^{i+r} W^a & V^b & Y^c & (-1)^{i+r} X^{ab} & (-1)^{i+r} Z^{ac} & (-1)^k S^{bc} & (-1)^{i+k+r} T^{abc} \\ V & (-1)^i X^a & Y^b & (-1)^k S^c & (-1)^i Z^{ab} & (-1)^{i+k} T^{ac} & (-1)^k U^{bc} & (-1)^{i+k} W^{abc} \\ Z & (-1)^r Y^a & (-1)^k T^b & (-1)^k W^b & (-1)^{k+r} S^{ab} & (-1)^{k+r} U^{ac} & (-1)^k X^{bc} & (-1)^{k+r} V^{abc} \\ Y & (-1)^{i+r} Z^a a & (-1)^k S^b & (-1)^k U^b & (-1)^{i+k+r} T^{ab} & (-1)^{i+k+r} W^{ac} & (-1)^k V^{bc} & (-1)^{i+k+r} X^{abc} \end{bmatrix}$$

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$$H_1 \equiv \mathcal{H}_1(T, S, W, X, U, V, Z, Y)^{a,b,c,i,k,r}$$

Block-structure-preserving equivalences

Proposition

$$H_1 \equiv \mathcal{H}_1(T, S, W, X, U, V, Z, Y)^{a,b,c,i,k,r}$$

$$\begin{aligned} H_2 &\equiv \mathcal{H}_2(T, S, U, V, W, X, Y, Z)^{a,b,c,i,j,k,r,s,t} \equiv \mathcal{H}_2(U, W, V, Y, X, Z, S, T)^{a,b,c,i,j,k,r,s,t} \\ &\equiv \mathcal{H}_2(V, X, Y, S, Z, T, U, W)^{a,b,c,i,j,k,r,s,t} \equiv \mathcal{H}_2(W, X, Z, T, Y, S, W, U)^{a,b,c,i,j,k,r,s,t} \\ &\equiv \mathcal{H}_2(X, V, Z, T, Y, S, W, U)^{a,b,c,i,j,k,r,s,t} \equiv \mathcal{H}_2(Y, Z, S, U, T, W, V, X)^{a,b,c,i,j,k,r,s,t} \\ &\equiv \mathcal{H}_2(Z, Y, T, W, S, U, X, V)^{a,b,c,i,j,k,r,s,t} \end{aligned}$$

$$H_3 \equiv \mathcal{H}_3(T, S, W, X, U, V, Z, Y)_{\alpha}^{a,b,c,i,j,k,r,s,t}$$

$$\begin{aligned} H_4 &\equiv \mathcal{H}_4(T, V, W, X, Y, S, Z, U)_{\varepsilon}^{a,b,c,i,r} \equiv \mathcal{H}_4(U, Z, V, Y, T, W, S, X)_{\varepsilon}^{a,b,c,i,r} \\ &\equiv \mathcal{H}_4(V, X, Y, S, Z, T, U, W)_{\varepsilon}^{a,b,c,i,r} \equiv \mathcal{H}_4(W, U, X, Z, V, Y, T, S)_{\varepsilon}^{a,b,c,i,r} \\ &\equiv \mathcal{H}_4(X, S, Z, T, U, V, W, Y)_{\varepsilon}^{a,b,c,i,r} \equiv \mathcal{H}_4(Y, W, S, U, X, Z, V, T)_{\varepsilon}^{a,b,c,i,r} \\ &\equiv \mathcal{H}_4(Z, Y, T, W, S, U, X, V)_{\varepsilon}^{a,b,c,i,r} \end{aligned}$$

Block-structure-preserving equivalences

So we know $H_1 \equiv \mathcal{H}_1(T, S, W, X, U, V, Z, Y)^{a,b,c,i,k,r}$ but can we do any better?

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with $(i, r) \in \{(0, 1), (1, 0), (1, 1)\}$.

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There are more block-structure-preserving equivalences arising from the specialisation of the "actions" a, b, c, and the parameters i, j, k, r, s, t and α, ε.

Control of eigenvalues

Recall

$$H_1 = \mathcal{H}_1(S, T, U, V, W, X, Y, Z)^{a,b,c,i,k,r} \qquad H_2 = \mathcal{H}_2(S, T, U, V, W, X, Y, Z)^{a,b,c,i,j,k,r,s,t}$$

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From the equation

$$H_i H_i^T = 8pI_{8p},$$

for $i = 1, 2, 3, 4$, it follows that

$$SS^T + \cdots + ZZ^T = 8pI_p.$$

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The gramians $SS^T, \ldots, ZZ^T$ commute in pairs and are simultaneously diagonalisable.

Control of eigenvalues

For $i = 1, \ldots, 8$ and $R \in \{S, \ldots, Z\}$, let

$$\lambda_{i,R}$$

denote the i-th eigenvalue of RR^T.

If $A \subset \{S, \ldots, Z\}$ then for $i = 1, \ldots, 8$ we have

$$\sum_{R \in A} \lambda_{i,R} \leq 8p. \tag{7}$$

These inequalities can help to trim the search spaces significantly.

Algorithm

This algorithm describes a method to classify all CHMs H_1, H_2, H_3, H_4 of order $8p$ with $p > 3$ prime up to equivalence.

Let $s, \dots, z$ the sums of the first rows of the blocks $S, \dots, Z$, respectively.

Input: a prime $p > 3$
Output: a list of all CHMs of order 8p, up to equivalence

initialise L as an empty list
determine all decompositions $\mathcal{D} = \{(s, \dots, z) \in \mathbb{Z}^8 \mid s^2 + \cdots + z^2 = 8p\}$
discard the element of $\mathcal{D}$ that produce equivalent matrices
for $(s, \dots, z) \in \mathcal{D}$ **do**
    construct $\mathcal{S}$ as the set of back-circulant matrices over ± 1 of order p with row
    sum s (that satisfy the eigenvalue constraint)
    similarly, construct $\mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$.
    for $(S, \dots, Z) \in \mathcal{S} \times \cdots \times \mathcal{Z}$ satisfying the eigenvalue constraints **do**
        construct H_1, H_2, H_3, H_4.
        if H_i is Hadamard and $H \notin L$ up to equivalence **then** add H_i to L.
return L.

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print Thank you!