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Fourth moment of random determinants

RICCOTA 2023, Rijeka

Definitions I:

X_{ij} i.i.d., $\qquad m_q = \mathbb{E}X_{ij}^q$,

$A = (X_{ij})_{n\times n} \qquad U = (X_{ij})_{n\times p}$,

$f_k(n) = \mathbb{E}(\det A)^k \qquad f_k(n,p) = \mathbb{E}(\det U^\top U)^{k/2}$,

$$F_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} f_k(n), \quad F_k(t,\omega) = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{t^p \omega^{n-p}(n-p)!}{n!p!} f_k(n,p).$$

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- WLOG $m_2 = 1$.

Previously known results

NRR's formula on fourth determinant moment

Theorem. (1954 | Nyquist H., Rice S. O., Riordan J.[1])

For any distribution of X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_4(t) = \frac{e^{t(m_4-3)}}{(1-t)^3}.$$

[1] Harry Nyquist, SO Rice, and J Riordan. "The distribution of random determinants". In: *Quarterly of Applied mathematics* 12.2 (1954), pp. 97–104

Previously known results

Dembo's formula on fourth Gram moment

Theorem. (1989 | Dembo A.[2])

For any distribution of X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_4(t,\omega) = \frac{e^{t(m_4-3)}}{(1-t)^2(1-\omega-t)}.$$

[2] Amir Dembo. "On random determinants". In: *Quarterly of applied mathematics* 47.2 (1989), pp. 185–195

Our results

General fourth determinant moment

Theorem. (5/2022 | B. D..[3])

For any distribution of X_{ij} with $m_1 = 0$ and $\mu_2 = 1$, $F_4(t)$ equals

$$\frac{e^{t(\mu_4-3)}}{(1-t)^3}\left[(1+m_1\mu_3 t)^4+6m_1^2 t\frac{(1+m_1\mu_3 t)^2}{1-t}+m_1^4 t\frac{1+7t+4t^2}{(1-t)^2}\right],$$

where $\mu_2 = m_2 - m_1^2$, $\mu_3 = m_3 - 3m_1 m_2 + 2m_1^3$ and $\mu_4 = m_4 - 4m_1 m_3 + 6m_1^2 m_2 - 3m_1^4$.

[3] Dominik Beck. "On the fourth moment of a random determinant". In: *arXiv preprint arXiv:2207.09311* (2022)

Our results

General fourth Gram moment

Theorem. (7/2022 | B. D.[4])

For any distribution of X_{ij} with $m_1 = 0$ and $\mu_2 = 1$ (μ_q as before),

$$F_4(t,\omega) = \frac{e^{t(\mu_4-3)}}{(1-t)^2(1-\omega-t)}\Bigg[(1+m_1\mu_3 t)^4 + \frac{6m_1^2 t\,(1+m_1\mu_3 t)^2}{1-t} + \frac{m_1^4 t\,(1+7t+4t^2)}{(1-t)^2} + \frac{\omega m_1^2 t}{1-\omega-t}\left(\frac{2\,(1+m_1\mu_3 t)^2}{1-t} + \frac{m_1^2\,(1+5t+2t^2)}{(1-t)^2}\right) + \frac{2t^2\omega^2 m_1^4}{(1-\omega-t)^2(1-t)^2}\Bigg].$$

[4] Dominik Beck. "On the fourth moment of a random determinant". In: *arXiv preprint arXiv:2207.09311* (2022)

Applications

Random geometry

V_d volume of random d-simplex picked from fixed d-simplex ($\mathbb{R}^d$) of unit content. For k even and $X_{ij} \sim \mathrm{Exp}(1)$, we have[5]

$$\mathbb{E} V_d^k = \left(\frac{d!}{(d+k)!} \right)^{d+1} f_k(d+1),$$

n	1	2	3	4	5
$f_4(n)$	24	960	51840	3511872	287953920

n	6	7	8
$f_4(n)$	27988001280	3181325414400	418846663065600

[5] William Reed. "Random points in a simplex". In: *Pacific Journal of Mathematics* 54.2 (1974), pp. 183–198

Definitions II:

$$Y_{ij} = X_{ij} - m_1, \qquad \mu_q = \mathbb{E} Y_{ij}^q,$$

$$B = (Y_{ij})_{n\times n} \qquad V = (Y_{ij})_{n\times p}$$

$$g_k(n) = \mathbb{E}(\det B)^k \qquad g_k(n,p) = \mathbb{E}(\det V^\top V)^{k/2}$$

$$G_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} g_k(n), \quad G_k(t,\omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{t^p \omega^{n-p}(n-p)!}{n!p!} g_k(n,p).$$

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- WLOG $\mu_2 = 1$.

Lemma.

For any distribution of X_{ij} with $\mu_2 = 1$,

$$G_k(t) = F_k(t)|_{m_q \to \mu_q}, \quad G_k(t, \omega) = F_k(t, \omega)|_{m_q \to \mu_q}, \quad q \geqslant 2.$$

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For any distribution of X_{ij} with $\mu_2 = 1$,

$$G_k(t) = F_k(t)|_{m_q \to \mu_q}, \quad G_k(t,\omega) = F_k(t,\omega)|_{m_q \to \mu_q}, \quad q \geqslant 2.$$

- $G_4(t) = \dfrac{e^{t(\mu_4 - 3)}}{(1-t)^3}$

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- $G_4(t) = \dfrac{e^{t(\mu_4 - 3)}}{(1-t)^3}$
- $G_4(t,\omega) = \dfrac{e^{t(\mu_4 - 3)}}{(1-t)^2(1-\omega-t)}$

Lemma (Matrix determinant lemma).

$$\det A = \det B + m_1 \sum_{ij} (-1)^{i+j} \det B_{ij}.$$

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Corollary. Denote $\pi \in T_n$ a permutation of order n and $\sigma \in T_n^\times$ a *marked permutation* formed from π by marking **at most one** of its numbers, then

$$\det A = \sum_{\pi \in T_n} \operatorname{sgn} \pi \prod_{i=1}^{n} X_{i\pi(i)} = \sum_{\sigma \in T_n^\times} \operatorname{sgn} \sigma \prod_{i=1}^{n} Y_{i\sigma(i)},$$

where $\operatorname{sgn} \sigma = \operatorname{sgn} \pi$ and $Y_{i\sigma(i)} = m_1$ if i is marked and $Y_{i\sigma(i)} = Y_{i\pi(i)}$ otherwise.

Marked tables

Definition. We say t is a *marked table* k by n if its rows are marked permutations. $T_{k,n}^{\times}$ is the set of all such tables. We define weight of its i-th column as $\mathbb{E}\prod_{j=0}^{k} Y_{i\sigma_j(i)}$ and the weight $w(t)$ of the whole table t as the product of weights of all of its columns. Also $\operatorname{sgn} t$ will be the product of signs of permutations of rows.

3	$\times$	1	4	5	2	7	8	9
3	2	1	9	4	6	$\times$	5	8
3	$\times$	1	9	4	2	7	5	8
3	2	1	4	5	6	7	8	9

Example: $t \in T_{4,9}^{\times}$ with $w(t) = m_1^3 \mu_3 \mu_4^2$

Proposition.

$$f_k(n) = \sum_{t \in T^{\times}_{k,n}} w(t) \operatorname{sgn} t.$$

Definition. Let $T^r_{k,n} \subseteq T^{\times}_{k,n}$ be the subset of those tables which have exactly r marks.

Definition. $f_k^{[r]}(n) = \sum_{t \in T^r_{k,n}} w(t) \operatorname{sgn} t$

Corollary. $f_k(n) = \sum_{r=0}^{k} f_k^{[r]}(n)$

Column types

Let a, b denote different numbers selected from $\{1, 2, 3, \ldots, n\}$. Up to permutation of rows, the only ways how the columns of 4 by n tables with nonzero weight could look like are the following:

Type:	4-column	2-column	$\times^1$-column	$\times^2$-column	$\times^4$-column
	$\begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix}$	$\begin{pmatrix} a \\ a \\ b \\ b \end{pmatrix}$	$\begin{pmatrix} \times \\ a \\ a \\ a \end{pmatrix}$	$\begin{pmatrix} \times \\ \times \\ a \\ a \end{pmatrix}$	$\begin{pmatrix} \times \\ \times \\ \times \\ \times \end{pmatrix}$
Weight:	μ_4	1	$m_1\mu_3$	m_1^2	m_1^4

1 Introduction

2 General fourth moment
- Preliminaries
- Marked tables
- Structure of marked tables
- Decomposition over even marked tables
- Covering technique

3 Gram fourth moment

4 Sixth moment (current work)

Definition. Let $S^r_{4,n} \subseteq T^r_{4,n}$ be the subset of those tables with nonzero weight which lack $\times^1$ columns.

Definition. $s_4^{[r]}(n) = \sum_{t \in S^r_{4,n}} w(t) \operatorname{sgn} t, \quad S_4^{[r]}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} s_4^{[r]}(n).$

Proposition.

$$F_4(t) = \sum_{r=0}^{4} (1 + m_1 \mu_3 t)^{4-r} S_4^{[r]}(t)$$
$$= (1 + m_1 \mu_3 t)^4 S_4^{[0]}(t) + (1 + m_1 \mu_3 t)^2 S_4^{[2]}(t) + S_4^{[4]}(t).$$

Proof. Creating $t \in T_{4,n}^r$ from $t' \in S_{4,n-s}^{r-s}$ by adding s $\times^1$-columns,

$$f_4^{[r]}(n) = \sum_{s=0}^{r} \binom{4-r+s}{s} \binom{n}{s}^2 s!^2 m_1^s \mu_3^s s_4^{[r-s]}(n-s)$$

3	$\times$	1	4	5	2	**7**	8	9		$\times$	1	4	5	2	8	9
3	2	1	9	4	6	$\times$	5	8	$\leftarrow$	2	1	9	4	6	5	8
3	$\times$	1	9	4	2	**7**	5	8		$\times$	1	9	4	2	5	8
$\times$	2	1	4	5	6	**7**	8	9		2	1	4	5	6	8	9

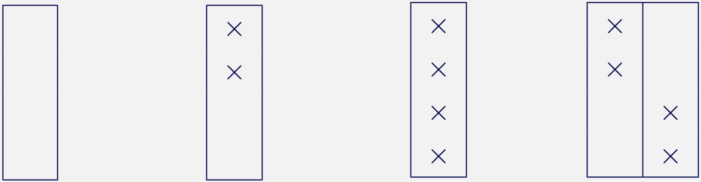
Example: $n = 9$, $r = 4$, $s = 2$

∎

Table types

Definition. We define tables $S_{4,n}^{r/s} \subseteq S_{4,n}^{r}$ such that their r marks occupy s columns. Accordingly, we define

$$s_4^{[r/s]}(n) = \sum_{t \in S_{4,n}^{r/s}} w(t)\,\mathrm{sgn}(t) \qquad \text{and} \qquad S_4^{[r/s]}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} s_4^{[r/s]}(n).$$



$S_{4,n}^{0}$ $S_{4,n}^{2}$ $S_{4,n}^{4/1}$ $S_{4,n}^{4/2}$

Proposition.

$$S_4^{[2]}(t) = m_1^2(6 - 2\mu_4)\frac{\partial S_4^{[0]}(t)}{\partial \mu_4} + 2m_1^2 t\,\frac{\partial S_4^{[0]}(t)}{\partial t}$$

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- $S_4^{[0]}(t) = G_4(t) = \dfrac{e^{t(\mu_4 - 3)}}{(1-t)^3}$

Corollary. $S_4^{[2]}(t) = \dfrac{6m_1^2 t e^{t(\mu_4-3)}}{(1-t)^4}$

Proof. Let $t' \in S^0_{4,n}$ have c of 4-columns and thus $n-c$ of 2-columns. Its weight is μ_4^c. From this t', we create $t \in S^2_{4,n}$ by covering two pairs of numbers in either 4-column or 2-column.

$$S^2_{4,n} \leftarrow S^0_{4,n}: \qquad \boxed{\begin{matrix}\times\\ \times\\ a\\ a\end{matrix}} \leftarrow \boxed{\begin{matrix}a\\ a\\ a\\ a\end{matrix}} \qquad \boxed{\begin{matrix}\times\\ \times\\ b\\ b\end{matrix}} \leftarrow \boxed{\begin{matrix}a\\ a\\ b\\ b\end{matrix}}$$

6 ways 2 ways

The contribution of t' to $\sum_{t \in S^{[2]}_{4,n}} w(t) \operatorname{sgn} t$ is then

$$6cm_1^2\mu_4^{c-1} + 2(n-c)m_1^2\mu_4^c = m_1^2(6-2\mu_4)\frac{\partial \mu_4^c}{\partial \mu_4} + 2m_1^2 n\mu_4^c. \quad \blacksquare$$

Proposition.

$$S_4^{[4/1]}(t) = m_1^4(1-\mu_4)\frac{\partial S_4^{[0]}(t)}{\partial \mu_4} + m_1^4 t\,\frac{\partial S_4^{[0]}(t)}{\partial t}$$

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Corollary. $S_4^{[4/1]}(t) = \dfrac{m_1^4 t(1+2t)}{(1-t)^4} e^{t(\mu_4-3)}$

Proof. Let $t' \in S^0_{4,n}$ have c of 4-columns and thus $n - c$ of 2-columns. Its weight is μ_4^c. From this t', we create $t \in S^{[4/1]}_{4,n}$ by covering all numbers in either 4-column or 2-column.

$$S^{4/1}_{4,n} \leftarrow S^0_{4,n} : \qquad \begin{pmatrix} \times \\ \times \\ \times \\ \times \end{pmatrix} \leftarrow \begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix} \qquad \begin{pmatrix} \times \\ \times \\ \times \\ \times \end{pmatrix} \leftarrow \begin{pmatrix} a \\ a \\ b \\ b \end{pmatrix}$$

The contribution of t' to $\sum_{t \in S^{[4/1]}_{4,n}} w(t) \operatorname{sgn} t$ is then

$$c m_1^4 \mu_4^{c-1} + (n - c) m_1^4 \mu_4^c = m_1^4 (1 - \mu_4) \frac{\partial \mu_4^c}{\partial \mu_4} + m_1^4 n \mu_4^c. \qquad \blacksquare$$

Proposition.

$$S_4^{[4/2]}(t) = (3 - \mu_4)\frac{\partial S_4^{[4/1]}(t)}{\partial \mu_4} + t\,\frac{\partial S_4^{[4/1]}(t)}{\partial t} - S_4^{[4/1]}(t)$$

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Corollary.

$$S_4^{[4]}(t) = S_4^{[4/1]}(t) + S_4^{[4/2]}(t) = \frac{m_1^4 t(1+7t+4t^2)}{(1-t)^5} e^{t(\mu_4-3)}$$

Proposition.

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Corollary. $F_4(t)$

Proof. Let $t' \in S_{4,n}^{[4/1]}$ have c of 4-columns and thus $n-c-1$ of 2-columns as now one column is a $\times^4$-column. The weight of t' is $m_1^4 \mu_4^c$. From this t', we create $t \in S_{4,n}^{[4/2]}$ by **swapping** its two $\times$ marks with pair of numbers in either 4-column or 2-column. By symmetry, each table in $S_{4,n}^{[4/2]}$ is counted twice.

$$S_{4,n}^{4/2} \leftarrow S_{4,n}^{4/1} : \quad \begin{array}{|c|c|} \times & a \\ \times & a \\ a & \times \\ a & \times \end{array} \leftarrow \begin{array}{|c|c|} \times & a \\ \times & a \\ \times & a \\ \times & a \end{array} \qquad \begin{array}{|c|c|} \times & a \\ \times & a \\ b & \times \\ b & \times \end{array} \leftarrow \begin{array}{|c|c|} \times & a \\ \times & a \\ \times & b \\ \times & b \end{array}$$

The contribution of t' to double of $\sum_{t \in S_{4,n}^{[4/2]}} w(t) \operatorname{sgn} t$ is then

$$6cm_1^4\mu_4^{c-1} + 2(n-c-1)m_1^4\mu_4^c = (6-2\mu_4)\frac{\partial(m_1^4\mu_4^c)}{\partial\mu_4} + 2nm_1^4\mu_4^c - 2m_1^4\mu_4^c. \qquad \blacksquare$$

Paired marked tables

Definition. We say t is a *paired marked table* k by n (k even) if for $i = 1, \ldots, k/2$, its $(2i-1)$-th and $2i$-th row are marked permutations of selection $C_i \subseteq \{1, 2, 3, \ldots n\}$. Similarly, $T^{\times}_{k,n,p}$ is the set of all such tables and $S^{\times}_{k,n,p}$ the subset of those lacking $\times^1$ columns and having nonzero weight. We define generating functions of those subsets accordingly.

Proposition.

$$f_k(n,p) = \sum_{t \in T^{\times}_{k,n,p}} w(t) \operatorname{sgn} t,$$

$$F_4(t,\omega) = \sum_{r=0}^{4} m_1^r (1 + m_1 \mu_3 t)^{4-r} S_4^{[r]}(t,\omega).$$

Proposition.

Per analogy, we must have for any distribution X_{ij} with $\mu_2 = 1$,

$$S_4^{[2]}(t,\omega) = m_1^2(6-2\mu_4)\frac{\partial S_4^{[0]}(t,\omega)}{\partial \mu_4} + 2m_1^2 t\,\frac{\partial S_4^{[0]}(t,\omega)}{\partial t},$$

$$S_4^{[4/1]}(t,\omega) = m_1^4(1-\mu_4)\frac{\partial S_4^{[0]}(t,\omega)}{\partial \mu_4} + m_1^4 t\,\frac{\partial S_4^{[0]}(t,\omega)}{\partial t},$$

$$S_4^{[4/2]}(t,\omega) = (3-\mu_4)\frac{\partial S_4^{[4/1]}(t,\omega)}{\partial \mu_4} + t\frac{\partial S_4^{[4/1]}(t,\omega)}{\partial t} - S_4^{[4/1]}(t,\omega),$$

$$S_4^{[4]}(t,\omega) = S_4^{[4/1]}(t,\omega) + S_4^{[4/2]}(t,\omega).$$

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$$S_4^{[4]}(t,\omega) = S_4^{[4/1]}(t,\omega) + S_4^{[4/2]}(t,\omega).$$

- $S_4^{[0]}(t,\omega) = G_4(t,\omega)$.

Corollary.

For any distribution of X_{ij} with $m_1 = 0$ and $\mu_2 = 1$ (μ_q as before),

$$F_4(t,\omega) = \frac{e^{t(\mu_4-3)}}{(1-t)^2(1-\omega-t)}\Bigg[(1+m_1\mu_3 t)^4 + \frac{6m_1^2 t\,(1+m_1\mu_3 t)^2}{1-t} + \frac{m_1^4 t\,(1+7t+4t^2)}{(1-t)^2} + \frac{\omega m_1^2 t}{1-\omega-t}\left(\frac{2\,(1+m_1\mu_3 t)^2}{1-t} + \frac{m_1^2\,(1+5t+2t^2)}{(1-t)^2}\right) + \frac{2t^2\omega^2 m_1^4}{(1-\omega-t)^2(1-t)^2}\Bigg].$$

Sixth moment

Theorem (12/2022 | B. D., Lv Zelin, Potechin Aaron[6])

For any disribution X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_6(t) = \frac{(1+m_3^2 t)^{10} e^{t\left(m_6 - 15m_4 - 10m_3^2 + 30\right)}}{48\,(1+3t-m_4 t)^{15}} \sum_{n=0}^{\infty} \frac{(1+n)(2+n)(4+n)!\,t^n}{(1+3t-m_4 t)^{3n}}.$$

[6] Dominik Beck, Zelin Lv, and Aaron Potechin. *The Sixth Moment of Random Determinants*. 2022. DOI: 10.48550/ARXIV.2206.11356. URL: https://arxiv.org/abs/2206.11356

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- **Corollary:** Explicit formula for $f_6(n)$ and its asymptotics.

[6] Dominik Beck, Zelin Lv, and Aaron Potechin. *The Sixth Moment of Random Determinants*. 2022. DOI: 10.48550/ARXIV.2206.11356. URL: https://arxiv.org/abs/2206.11356

Gram Sixth moment

Theorem (6/2023 | B. D.)

For any disribution X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_6(t,\omega) = \frac{(1+m_3^2 t)^{10} e^{t\left(m_6 - 15 m_4 - 10 m_3^2 + 30\right)}}{48\,(1+3t-m_4 t)^{15}} \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{t^p \omega^{n-p}(n+2)!(n+4)!}{p!(n-p+2)!(n-p+4)!\,(1+3t-m_4 t)^{3n}}.$$

Open questions

For general distribution X_{ij} with $m_1 \neq 0$, determine

- $F_6(t)$
- $F_6(t, \omega)$

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For general distribution X_{ij} with $m_1 \neq 0$, determine

- $F_6(t)$
- $F_6(t, \omega)$

As of 28 June, we know only the special case $m_1 \neq 0$, $\mu_3 = 0$.

Thank you for your attention!

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