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Fourth moment of random determinants

RICCOTA 2023, Rijeka

- 1 Introduction
- 2 General fourth moment
- 3 Gram fourth moment
- 4 Sixth moment (current work)

Definitions I:

 X_{ij} i.i.d.,

$$m_q = \mathbb{E}X_{ij}^q,$$

$$A = (X_{ij})_{n \times n}$$

$$U = (X_{ij})_{n \times p},$$

$$f_k(n) = \mathbb{E}(\det A)^k$$

$$f_k(n, p) = \mathbb{E}(\det U^\top U)^{k/2},$$

$$F_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} f_k(n), \quad F_k(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^n \frac{t^p \omega^{n-p} (n-p)!}{n! p!} f_k(n, p).$$

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- WLOG $m_2 = 1$.

Previously known results

NRR's formula on fourth determinant moment

Theorem. (1954 | Nyquist H., Rice S. O., Riordan J.¹)

For any distribution of X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_4(t) = \frac{e^{t(m_4-3)}}{(1-t)^3}.$$

¹Harry Nyquist, SO Rice, and J Riordan. "The distribution of random determinants". In: *Quarterly of Applied mathematics* 12.2 (1954), pp. 97–104

Previously known results

Dembo's formula on fourth Gram moment

Theorem. (1989 | Dembo A.²)

For any distribution of X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_4(t, \omega) = \frac{e^{t(m_4-3)}}{(1-t)^2(1-\omega-t)}.$$

²Amir Dembo. "On random determinants". In: *Quarterly of applied mathematics* 47.2 (1989), pp. 185–195

Our results

General fourth determinant moment

Theorem. (5/2022 | B. D.³)

For any distribution of X_{ij} with $m_1 = 0$ and $\mu_2 = 1$, $F_4(t)$ equals

$$\frac{e^{t(\mu_4-3)}}{(1-t)^3} \left[(1+m_1\mu_3t)^4 + 6m_1^2t \frac{(1+m_1\mu_3t)^2}{1-t} + m_1^4t \frac{1+7t+4t^2}{(1-t)^2} \right],$$

where $\mu_2 = m_2 - m_1^2$, $\mu_3 = m_3 - 3m_1m_2 + 2m_1^3$ and $\mu_4 = m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4$.

³Dominik Beck. "On the fourth moment of a random determinant". In: *arXiv preprint arXiv:2207.09311* (2022)

Our results

General fourth Gram moment

Theorem. (7/2022 | B. D.⁴)

For any distribution of X_{ij} with $m_1 = 0$ and $\mu_2 = 1$ (μ_q as before),

$$F_4(t, \omega) = \frac{e^{t(\mu_4-3)}}{(1-t)^2(1-\omega-t)} \left[(1+m_1\mu_3t)^4 + \frac{6m_1^2t(1+m_1\mu_3t)^2}{1-t} \right. \\ \left. + \frac{m_1^4t(1+7t+4t^2)}{(1-t)^2} + \frac{\omega m_1^2t}{1-\omega-t} \left(\frac{2(1+m_1\mu_3t)^2}{1-t} \right. \right. \\ \left. \left. + \frac{m_1^2(1+5t+2t^2)}{(1-t)^2} \right) + \frac{2t^2\omega^2m_1^4}{(1-\omega-t)^2(1-t)^2} \right].$$

⁴Dominik Beck. "On the fourth moment of a random determinant". In: *arXiv preprint arXiv:2207.09311* (2022)

Applications

Random geometry

V_d volume of random d -simplex picked from fixed d -simplex (\mathbb{R}^d) of unit content. For k even and $X_{ij} \sim \text{Exp}(1)$, we have⁵

$$\mathbb{E}V_d^k = \left(\frac{d!}{(d+k)!} \right)^{d+1} f_k(d+1),$$

n	1	2	3	4	5
$f_4(n)$	24	960	51840	3511872	287953920
n	6		7		8
$f_4(n)$	27988001280		3181325414400		418846663065600

⁵William Reed. "Random points in a simplex". In: *Pacific Journal of Mathematics* 54.2 (1974), pp. 183–198

1 Introduction

2 General fourth moment

- Preliminaries
- Marked tables
- Structure of marked tables
- Decomposition over even marked tables
- Covering technique

3 Gram fourth moment

4 Sixth moment (current work)

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Definitions II:

$$Y_{ij} = X_{ij} - m_1,$$

$$\mu_q = \mathbb{E} Y_{ij}^q,$$

$$B = (Y_{ij})_{n \times n}$$

$$V = (Y_{ij})_{n \times p}$$

$$g_k(n) = \mathbb{E}(\det B)^k$$

$$g_k(n, p) = \mathbb{E}(\det V^T V)^{k/2}$$

$$G_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} g_k(n), \quad G_k(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^n \frac{t^p \omega^{n-p} (n-p)!}{n! p!} g_k(n, p).$$

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- WLOG $\mu_2 = 1$.

Lemma.

For any distribution of X_{ij} with $\mu_2 = 1$,

$$G_k(t) = F_k(t)|_{m_q \rightarrow \mu_q}, \quad G_k(t, \omega) = F_k(t, \omega)|_{m_q \rightarrow \mu_q}, \quad q \geq 2.$$

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■ $G_4(t) = \frac{e^{t(\mu_4-3)}}{(1-t)^3}$

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$$\blacksquare G_4(t) = \frac{e^{t(\mu_4-3)}}{(1-t)^3}$$

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Lemma (Matrix determinant lemma).

$$\det A = \det B + m_1 \sum_{ij} (-1)^{i+j} \det B_{ij}.$$

Lemma (Matrix determinant lemma).

$$\det A = \det B + m_1 \sum_{ij} (-1)^{i+j} \det B_{ij}.$$

Corollary. Denote $\pi \in T_n$ a permutation of order n and $\sigma \in T_n^\times$ a *marked permutation* formed from π by marking **at most one** of its numbers, then

$$\det A = \sum_{\pi \in T_n} \operatorname{sgn} \pi \prod_{i=1}^n X_{i\pi(i)} = \sum_{\sigma \in T_n^\times} \operatorname{sgn} \sigma \prod_{i=1}^n Y_{i\sigma(i)},$$

where $\operatorname{sgn} \sigma = \operatorname{sgn} \pi$ and $Y_{i\sigma(i)} = m_1$ if i is marked and $Y_{i\sigma(i)} = Y_{i\pi(i)}$ otherwise.

Marked tables

Definition. We say t is a *marked table* k by n if its rows are marked permutations. $T_{k,n}^\times$ is the set of all such tables. We define weight of its i -th column as $\mathbb{E} \prod_{j=0}^k Y_{i\sigma_j(i)}$ and the weight $w(t)$ of the whole table t as the product of weights of all of its columns. Also $\text{sgn } t$ will be the product of signs of permutations of rows.

3	×	1	4	5	2	7	8	9
3	2	1	9	4	6	×	5	8
3	×	1	9	4	2	7	5	8
3	2	1	4	5	6	7	8	9

Example: $t \in T_{4,9}^\times$ with $w(t) = m_1^3 \mu_3 \mu_4^2$

Proposition.

$$f_k(n) = \sum_{t \in T_{k,n}^{\times}} w(t) \operatorname{sgn} t.$$

Definition. Let $T_{k,n}^r \subseteq T_{k,n}^{\times}$ be the subset of those tables which have exactly r marks.

Definition. $f_k^{[r]}(n) = \sum_{t \in T_{k,n}^r} w(t) \operatorname{sgn} t$

Corollary. $f_k(n) = \sum_{r=0}^k f_k^{[r]}(n)$

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Column types

Let a, b denote different numbers selected from $\{1, 2, 3, \dots, n\}$. Up to permutation of rows, the only ways how the columns of 4 by n tables with nonzero weight could look like are the following:

Type: 4-column 2-column \times^1 -column \times^2 -column \times^4 -column

$$\begin{array}{|c|} \hline a \\ \hline a \\ \hline a \\ \hline a \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline a \\ \hline a \\ \hline b \\ \hline b \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline \times \\ \hline a \\ \hline a \\ \hline a \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline \times \\ \hline \times \\ \hline a \\ \hline a \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \times \\ \hline \times \\ \hline \end{array}$$

Weight:

 μ_4
 1
 $m_1 \mu_3$
 m_1^2
 m_1^4

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Definition. Let $S_{4,n}^r \subseteq T_{4,n}^r$ be the subset of those tables with nonzero weight which lack \times^1 columns.

Definition. $s_4^{[r]}(n) = \sum_{t \in S_{4,n}^r} w(t) \operatorname{sgn} t$, $S_4^{[r]}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} s_4^{[r]}(n)$.

Proposition.

$$\begin{aligned} F_4(t) &= \sum_{r=0}^4 (1 + m_1 \mu_3 t)^{4-r} S_4^{[r]}(t) \\ &= (1 + m_1 \mu_3 t)^4 S_4^{[0]}(t) + (1 + m_1 \mu_3 t)^2 S_4^{[2]}(t) + S_4^{[4]}(t). \end{aligned}$$

Proof. Creating $t \in T_{4,n}^r$ from $t' \in S_{4,n-s}^{r-s}$ by adding $s \times 1$ -columns,

$$f_4^{[r]}(n) = \sum_{s=0}^r \binom{4-r+s}{s} \binom{n}{s}^2 s!^2 m_1^s \mu_3^s s_4^{[r-s]}(n-s)$$

3	×	1	4	5	2	7	8	9
3	2	1	9	4	6	×	5	8
3	×	1	9	4	2	7	5	8
×	2	1	4	5	6	7	8	9

←

×	1	4	5	2	8	9
2	1	9	4	6	5	8
×	1	9	4	2	5	8
2	1	4	5	6	8	9

Example: $n = 9$, $r = 4$, $s = 2$



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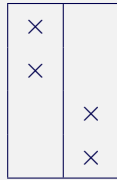
Table types

Definition. We define tables $S_{4,n}^{r/s} \subseteq S_{4,n}^r$ such that their r marks occupy s columns. Accordingly, we define

$$s_4^{[r/s]}(n) = \sum_{t \in S_{4,n}^{r/s}} w(t) \operatorname{sgn}(t) \quad \text{and} \quad S_4^{[r/s]}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} s_4^{[r/s]}(n).$$


 $S_{4,n}^0$

 $S_{4,n}^2$

 $S_{4,n}^{4/1}$

 $S_{4,n}^{4/2}$

Proposition.

$$S_4^{[2]}(t) = m_1^2(6 - 2\mu_4) \frac{\partial S_4^{[0]}(t)}{\partial \mu_4} + 2m_1^2 t \frac{\partial S_4^{[0]}(t)}{\partial t}$$

Proposition.

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- $S_4^{[0]}(t) = G_4(t) = \frac{e^{t(\mu_4-3)}}{(1-t)^3}$

Proposition.

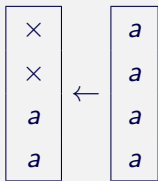
$$S_4^{[2]}(t) = m_1^2(6 - 2\mu_4) \frac{\partial S_4^{[0]}(t)}{\partial \mu_4} + 2m_1^2 t \frac{\partial S_4^{[0]}(t)}{\partial t}$$

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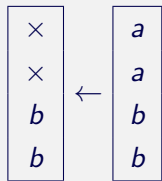
$$\text{Corollary. } S_4^{[2]}(t) = \frac{6m_1^2 t e^{t(\mu_4-3)}}{(1-t)^4}$$

Proof. Let $t' \in S_{4,n}^0$ have c of 4-columns and thus $n - c$ of 2-columns. Its weight is μ_4^c . From this t' , we create $t \in S_{4,n}^2$ by covering two pairs of numbers in either 4-column or 2-column.

$$S_{4,n}^2 \leftarrow S_{4,n}^0 :$$



6 ways



2 ways

The contribution of t' to $\sum_{t \in S_{4,n}^{[2]}} w(t) \operatorname{sgn} t$ is then

$$6cm_1^2 \mu_4^{c-1} + 2(n-c)m_1^2 \mu_4^c = m_1^2 (6 - 2\mu_4) \frac{\partial \mu_4^c}{\partial \mu_4} + 2m_1^2 n \mu_4^c. \quad \blacksquare$$

Proposition.

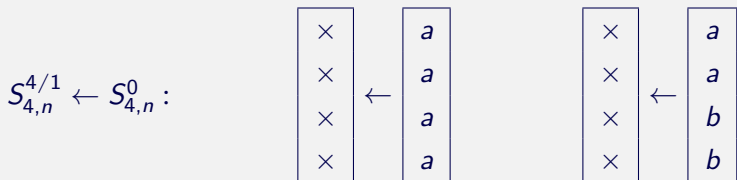
$$S_4^{[4/1]}(t) = m_1^4(1 - \mu_4) \frac{\partial S_4^{[0]}(t)}{\partial \mu_4} + m_1^4 t \frac{\partial S_4^{[0]}(t)}{\partial t}$$

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Corollary. $S_4^{[4/1]}(t) = \frac{m_1^4 t(1 + 2t)}{(1 - t)^4} e^{t(\mu_4 - 3)}$

Proof. Let $t' \in S_{4,n}^0$ have c of 4-columns and thus $n - c$ of 2-columns. Its weight is μ_4^c . From this t' , we create $t \in S_{4,n}^{[4/1]}$ by covering all numbers in either 4-column or 2-column.



The contribution of t' to $\sum_{t \in S_{4,n}^{[4/1]}} w(t) \operatorname{sgn} t$ is then

$$cm_1^4 \mu_4^{c-1} + (n-c)m_1^4 \mu_4^c = m_1^4 (1 - \mu_4) \frac{\partial \mu_4^c}{\partial \mu_4} + m_1^4 n \mu_4^c. \quad \blacksquare$$

Proposition.

$$S_4^{[4/2]}(t) = (3 - \mu_4) \frac{\partial S_4^{[4/1]}(t)}{\partial \mu_4} + t \frac{\partial S_4^{[4/1]}(t)}{\partial t} - S_4^{[4/1]}(t)$$

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Corollary. $S_4^{[4/2]}(t) = \frac{6m_1^4 t^2 (1+t)}{(1-t)^5} e^{t(\mu_4-3)}$

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Corollary. $S_4^{[4/2]}(t) = \frac{6m_1^4 t^2 (1+t)}{(1-t)^5} e^{t(\mu_4-3)}$

Corollary.

$$S_4^{[4]}(t) = S_4^{[4/1]}(t) + S_4^{[4/2]}(t) = \frac{m_1^4 t (1 + 7t + 4t^2)}{(1-t)^5} e^{t(\mu_4-3)}$$

Proposition.

$$S_4^{[4/2]}(t) = (3 - \mu_4) \frac{\partial S_4^{[4/1]}(t)}{\partial \mu_4} + t \frac{\partial S_4^{[4/1]}(t)}{\partial t} - S_4^{[4/1]}(t)$$

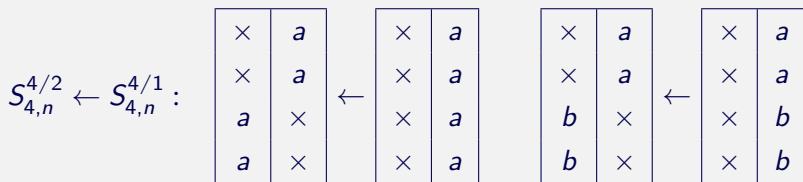
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Corollary. $F_4(t)$

Proof. Let $t' \in S_{4,n}^{[4/1]}$ have c of 4-columns and thus $n - c - 1$ of 2-columns as now one column is a \times^4 -column. The weight of t' is $m_1^4 \mu_4^c$. From this t' , we create $t \in S_{4,n}^{[4/2]}$ by **swapping** its two \times marks with pair of numbers in either 4-column or 2-column. By symmetry, each table in $S_{4,n}^{[4/2]}$ is counted twice.



The contribution of t' to double of $\sum_{t \in S_{4,n}^{[4/2]}} w(t) \operatorname{sgn} t$ is then

$$6cm_1^4 \mu_4^{c-1} + 2(n-c-1)m_1^4 \mu_4^c = (6-2\mu_4) \frac{\partial(m_1^4 \mu_4^c)}{\partial \mu_4} + 2nm_1^4 \mu_4^c - 2m_1^4 \mu_4^c. \blacksquare$$

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Paired marked tables

Definition. We say t is a *paired marked table* k by n (k even) if for $i = 1, \dots, k/2$, its $(2i - 1)$ -th and $2i$ -th row are marked permutations of selection $C_i \subseteq \{1, 2, 3, \dots, n\}$. Similarly, $T_{k,n,p}^\times$ is the set of all such tables and $S_{k,n,p}^\times$ the subset of those lacking \times^1 columns and having nonzero weight. We define generating functions of those subsets accordingly.

Proposition.

$$f_k(n, p) = \sum_{t \in T_{k,n,p}^\times} w(t) \operatorname{sgn} t,$$

$$F_4(t, \omega) = \sum_{r=0}^4 m_1^r (1 + m_1 \mu_3 t)^{4-r} S_4^{[r]}(t, \omega).$$

Proposition.

Per analogy, we must have for any distribution X_{ij} with $\mu_2 = 1$,

$$S_4^{[2]}(t, \omega) = m_1^2(6 - 2\mu_4) \frac{\partial S_4^{[0]}(t, \omega)}{\partial \mu_4} + 2m_1^2 t \frac{\partial S_4^{[0]}(t, \omega)}{\partial t},$$

$$S_4^{[4/1]}(t, \omega) = m_1^4(1 - \mu_4) \frac{\partial S_4^{[0]}(t, \omega)}{\partial \mu_4} + m_1^4 t \frac{\partial S_4^{[0]}(t, \omega)}{\partial t},$$

$$S_4^{[4/2]}(t, \omega) = (3 - \mu_4) \frac{\partial S_4^{[4/1]}(t, \omega)}{\partial \mu_4} + t \frac{\partial S_4^{[4/1]}(t, \omega)}{\partial t} - S_4^{[4/1]}(t, \omega),$$

$$S_4^{[4]}(t, \omega) = S_4^{[4/1]}(t, \omega) + S_4^{[4/2]}(t, \omega).$$

Proposition.

Per analogy, we must have for any distribution X_{ij} with $\mu_2 = 1$,

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$$S_4^{[4/1]}(t, \omega) = m_1^4(1 - \mu_4) \frac{\partial S_4^{[0]}(t, \omega)}{\partial \mu_4} + m_1^4 t \frac{\partial S_4^{[0]}(t, \omega)}{\partial t},$$

$$S_4^{[4/2]}(t, \omega) = (3 - \mu_4) \frac{\partial S_4^{[4/1]}(t, \omega)}{\partial \mu_4} + t \frac{\partial S_4^{[4/1]}(t, \omega)}{\partial t} - S_4^{[4/1]}(t, \omega),$$

$$S_4^{[4]}(t, \omega) = S_4^{[4/1]}(t, \omega) + S_4^{[4/2]}(t, \omega).$$

■ $S_4^{[0]}(t, \omega) = G_4(t, \omega).$

Corollary.

For any distribution of X_{ij} with $m_1 = 0$ and $\mu_2 = 1$ (μ_q as before),

$$F_4(t, \omega) = \frac{e^{t(\mu_4-3)}}{(1-t)^2(1-\omega-t)} \left[(1+m_1\mu_3t)^4 + \frac{6m_1^2t(1+m_1\mu_3t)^2}{1-t} \right. \\ \left. + \frac{m_1^4t(1+7t+4t^2)}{(1-t)^2} + \frac{\omega m_1^2t}{1-\omega-t} \left(\frac{2(1+m_1\mu_3t)^2}{1-t} \right. \right. \\ \left. \left. + \frac{m_1^2(1+5t+2t^2)}{(1-t)^2} \right) + \frac{2t^2\omega^2m_1^4}{(1-\omega-t)^2(1-t)^2} \right].$$

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Sixth moment

Theorem (12/2022 | B. D., Lv Zelin, Potechin Aaron⁶)

For any distribution X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_6(t) = \frac{(1+m_3^2 t)^{10} e^{t(m_6-15m_4-10m_3^2+30)}}{48(1+3t-m_4 t)^{15}} \sum_{n=0}^{\infty} \frac{(1+n)(2+n)(4+n)! t^n}{(1+3t-m_4 t)^{3n}}.$$

⁶Dominik Beck, Zelin Lv, and Aaron Potechin. *The Sixth Moment of Random Determinants*. 2022. DOI: 10.48550/ARXIV.2206.11356. URL: <https://arxiv.org/abs/2206.11356>

Sixth moment

Theorem (12/2022 | B. D., Lv Zelin, Potechin Aaron⁶)

For any distribution X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_6(t) = \frac{(1+m_3^2 t)^{10} e^{t(m_6-15m_4-10m_3^2+30)}}{48(1+3t-m_4 t)^{15}} \sum_{n=0}^{\infty} \frac{(1+n)(2+n)(4+n)! t^n}{(1+3t-m_4 t)^{3n}}.$$

- **Corollary:** Explicit formula for $f_6(n)$ and its asymptotics.

⁶Dominik Beck, Zelin Lv, and Aaron Potechin. *The Sixth Moment of Random Determinants*. 2022. DOI: 10.48550/ARXIV.2206.11356. URL: <https://arxiv.org/abs/2206.11356>

Gram Sixth moment

Theorem (6/2023 | B. D.)

For any distribution X_{ij} with $m_1 = 0$ and $m_2 = 1$,

$$F_6(t, \omega) = \frac{(1+m_3^2 t)^{10} e^{t(m_6-15m_4-10m_3^2+30)}}{48(1+3t-m_4 t)^{15}}$$

$$\sum_{n=0}^{\infty} \sum_{p=0}^n \frac{t^p \omega^{n-p} (n+2)!(n+4)!}{p!(n-p+2)!(n-p+4)!(1+3t-m_4 t)^{3n}}.$$

Open questions

For general distribution X_{ij} with $m_1 \neq 0$, determine

- $F_6(t)$
- $F_6(t, \omega)$

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- $F_6(t)$
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As of 28 June, we know only the special case $m_1 \neq 0, \mu_3 = 0$.

Thank you for your attention!

- [NRR54] Harry Nyquist, SO Rice, and J Riordan. “The distribution of random determinants”. In: *Quarterly of Applied mathematics* 12.2 (1954), pp. 97–104.
- [Ree74] William Reed. “Random points in a simplex”. In: *Pacific Journal of Mathematics* 54.2 (1974), pp. 183–198.
- [Dem89] Amir Dembo. “On random determinants”. In: *Quarterly of applied mathematics* 47.2 (1989), pp. 185–195.
- [Bec22] Dominik Beck. “On the fourth moment of a random determinant”. In: *arXiv preprint arXiv:2207.09311* (2022).
- [BLP22] Dominik Beck, Zelin Lv, and Aaron Potechin. *The Sixth Moment of Random Determinants*. 2022. DOI: 10.48550/ARXIV.2206.11356. URL: <https://arxiv.org/abs/2206.11356>.