Segre's theorem on ovals in Desarguesian projective planes

A nicer approach ?

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RICCOTA, 2023.

Motivation

Segre's theorem on ovals in projective spaces is an ingenious result from the mid-twentieth century which requires surprisingly little background to prove.

Theorem

The points of a maximal oval in a finite projective plane of odd characteristic satisfy a polynomial equation of degree 2.

While the theorem is well known, the aim of this talk is to provide a more approachable proof, that in addition to the lemma of the tangents, only uses the result of Desargues and some basic line constructions.
Segre's proof also requires a result of Qvist.

Preliminaries

A projective plane consists of points, lines and an incidence relation relating points and lines which obey the following:

- There is a unique line incident with any two distinct points.
- Any two distinct lines are incident with a unique point.
- There exist four points, no three incident with a line.

In addition, for a projective plane P, of order q:

- Given any point of P there are exactly $q+1$ lines incident,
- Given any line of P there are exactly $q+1$ points incident.

It *can* at times be helpful to think of the real projective plane for intuition.

Our approach

- Start with conics, since they are easy to motivate and understand.
- Definition of an oval
- Observe that a conic in a projective plane of odd order is a maximal oval, then prove Segre's theorem which is the converse of this.

Examples

Definition

A *conic* in projective space is the locus of points of a homogeneous polynomial of degree 2. A conic is said to be *non degenerate* if it is non-empty and does not contain an entire projective line.

Proposition

A non degenerate conic in $PG_2(\mathbb{F}_q)$ contains $q+1$ points and meets a line in at most two points.

We illustrate working in a finite projective plane with an example:

Example

Consider the conic $F(x, y, z) = x^2 - yz$, over a field with at least 3 elements. Let us compute the tangent at $[1, 1, 1]$ and some additional points on the curve.

- Thankfully (and surprisingly) the normal vector to F is given by $\nabla F = [2x : -z : -y]$ hence a tangent line will be a vector orthogonal to this.
- For example $p = [1 : 1 : 1]$ is in the variety of F, the normal vector here is $[2 : -1 : -1]$ which is orthogonal to, for example $[0 : 1 : -1]$. So that $T_p = [1 : 1 + t : 1 - t]$ is tangent. We can verify this by showing that:

$$1 - (1 + t)(1 - t) = t^2 = 0,$$

has a unique solution, hence T_p intersects the variety at only p.

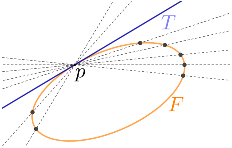


Figure: Something to visualise.

- We can identify more points on the conic by seeing were lines that contain p also meet the conic. Any line that contains p can be expressed as $[1+\alpha t : 1+\beta t : 1+\gamma t]$.
- For example the line for which $\alpha=\beta=1, \gamma=0$ will intersect the conic when,

$$(1+t)^2-(1+t)=0, \;\; t=0,-1,$$

so that $[0:0:1]$ is on F. We could if needed find all points in similar fashion.

A moments thought or thinking back to the real projective plane model should show you that the lines corresponding to the triples α, β, γ and α', β', γ' are the same if and only if

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix},$$

is invertible.

Definition

An *oval* in a projective plane is a subset of the points of the plane meeting no line in more than two points.

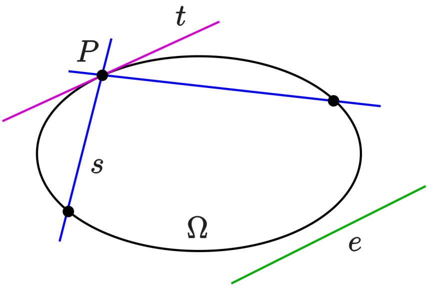


Figure: Probably a conic.

Proposition

An oval in a projective plane of order q contains at most $q+2$ points when q is even, and at most $q+1$ when q is odd.
A conic in a projective plane of odd order is a maximal oval.

Theorem (Desargues)

Let $P = \{p_1, p_2, p_3\}$ and $S = \{s_1, s_2, s_3\}$ be triangles in perspective in a projective plane. Denote by x_{ij} the intersection of the congruent sides $|p_i p_j|$ and $|s_i s_j|$ of the two triangles.
Then the points x_{12}, x_{13}, and x_{23} are collinear.

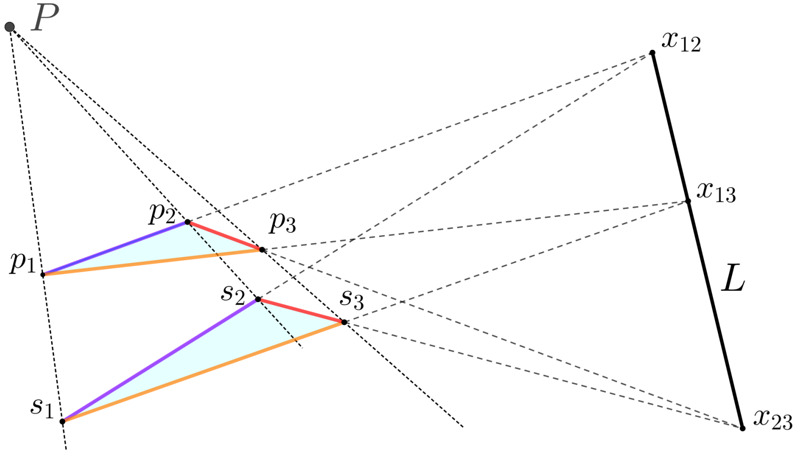


Figure: Desargues configuration.

Lemma of the tangents

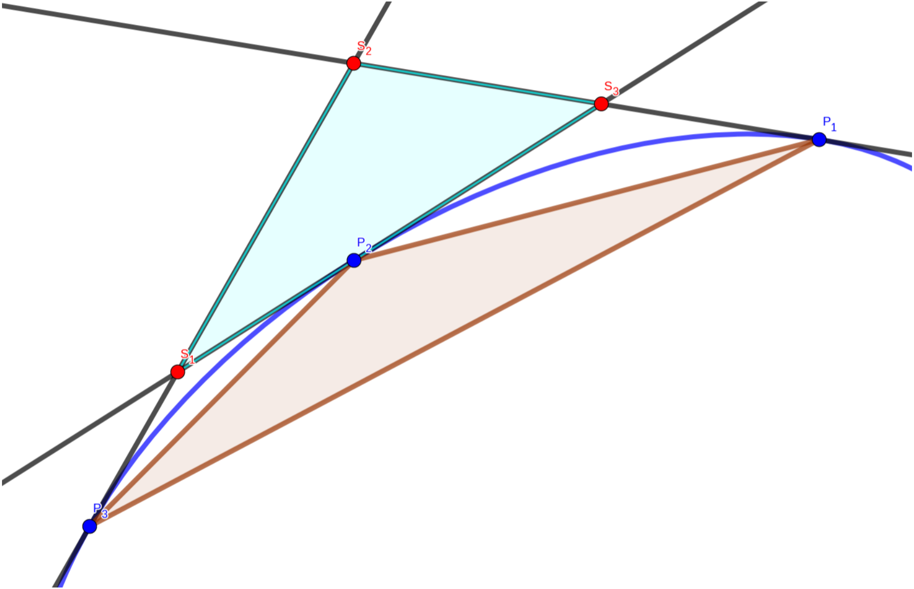
Before proceeding we need one more lemma (of the tangents), which we will just quote.

Lemma (Lemma of the tangents)

Let p_1, p_2, p_3 be the three distinct points on an oval in $PG_2(\mathbb{F}_q)$ where q is an odd prime power. Define s_i to be the intersection point of the tangents to the oval at p_{i+1} and p_{i+2} with subscripts modulo 3. The triangles $P = \{p_1, p_2, p_3\}$ and $S = \{s_1, s_2, s_3\}$ are then in perspective.

A simple picture will make this clearer.

Lemma of the tangents



Notes on Segre's paper

Before we prove Segre's theorem we remark on the differences between his approach and ours,

- The paper makes use of the Theorem of Qvist **&** Desargues
- While on the face shorter, there are some constructions and algebraic terms that to our eyes were not clear.
- Our overall idea and approach is similar but we do feel that ours is more readable.

Theorem (Segre)

The points of a maximal oval in a finite projective plane of odd characteristic satisfy a polynomial equation of degree 2.

Our basic idea will be to pick points on the oval compute the tangents at these points, then use the various triples along with the lemma of the tangents and Desargues to show that the points must obey a polynomial of degree 2.

- Choose a triangle $P = \{p_1, p_2, p_3\}$ on the oval. We are free to choose coordinates so that $p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1]$. Next we wish to construct lines incident to these points.
- For example the line joining p_1 to p_3 is of the form $[1, 0, \alpha]$ and has 'slope' $\frac{0}{\alpha} = 0$ we denote this line by $L_{p_1}(0)$.

- So that $L_{p_1}(\alpha)$ describes all q lines through p_1 with the remaining line at infinity, $L_{p_1}(\infty)$. Somewhere between $\alpha = 0$ and ∞ lies the tangent line to the oval, say $L_{p_1}(k_1)$. In similar fashion recover the tangent lines $L_{p_2}(k_2)$ and $L_{p_3}(k_3)$.
- We know that the k_i must satisfy, $k_1 k_2 k_3 = -1$. So up to projective equivalence we can choose $k_1 = k_2 = k_3 = -1$.
- With this in mind we now have the points p_i and their tangent lines:

p_1	p_2	p_3
$x_2 = -x_3$	$x_3 = -x_1$	$x_1 = -x_2$

- Let $c = [c_1 : c_2 : c_3]$ be a point on the oval distinct from P. Let $b_1x_1 + b_2x_2 + b_3x_3 = 0$ be the unique tangent to the oval at c. A small amount of consideration shows that $c_i \neq 0$ and $b_j \neq 0$.
- Consider the triangle $\{c, p_2, p_3\}$ which by the Lemma of the tangents is in perspective to the triangle formed by the tangents:

$$b_1x_1 + b_2x_2 + b_3x_3 = 0,\ x_3 = -x_1,\ x_1 = -x_2.$$

- By Desargues these triangles will be in perspective from a line. Again extending the lines as needed and setting parametric equations equal yield three collinear points, which imply that the determinant of the following matrix must vanish:

$$\begin{pmatrix} 0 & b_3 & -b_2 \\ c_1 & -c_1 & c_3 \\ c_1 & c_2 & -c_1 \end{pmatrix}$$

- We can repeat the above for the triangles $\{c, p_1, p_2\}$ and $\{c, p_1, p_3\}$ and using that we know the equation of the line that contains c.
- All of these together give the following relations:

$$\begin{aligned} b_3(c_1 + c_3) &= b_2(c_1 + c_2) \\ b_3(c_2 + c_3) &= b_(c_1 + c_2), \\ b_1(c_1 + c_3) &= b_2(c_2 + c_3), \\ (c_1 + c_2)(b_1 c_1 + b_2 c_2 + b_3 c_3) &= 0 \end{aligned}$$

- Finally, observing that the $b_i \neq 0$ and the odd characteristic, allow us to define a relation on the c_i. These being arbitrary allow us to conclude that the points of the oval obey, the equation of a conic.

Thank you.

Motivation for the talk/paper from a book by S.T. Dougherty, *Combinatorics and Finite Geometry (Springer Undergraduate Mathematics Series).*