Rijeka
Conference on
Combinatorial
Objects and
their Applications

Meet my favorite net

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HEFFTER ARRAYS

Given G of order $2mn+1$, with $3 \leq m \leq n$, a Heffter array $H(m,n)$ over G is a $m \times n$ matrix M with elements in G s.t.:

- if L is the set of elements of M, then $\{1,-1\} \cdot L = G \setminus \{0\}$
- each row is **zero-sum**;
- each column is **zero-sum**.

Example. An $H(3,3)$ over $\mathbb{Z}_{19}$:

$$\begin{bmatrix} 1 & 3 & -4 \\ 7 & 2 & -9 \\ -8 & -5 & -6 \end{bmatrix}$$

Existence:

cyclic case completely solved (no exception)

[Archdeacon - Boothby - Dinitz, JCD 2017]

elementary abelian case completely solved (no exception)

[MB, to appear in "Stinson66"]

→ main tool: RANK-ONE Heffter arrays

A very recent survey on the arXiv:

[A. Pasotti, J.H. Dinitz, A survey of Heffter arrays]

RANK-ONE HEFFTER ARRAYS

Let $q = 2mn + 1$ be a prime power.

$$\boxed{\text{rank-one } H(m,n) = H(m,n) \text{ over } \mathbb{F}_q \text{ whose rank is } 1}$$

(every row is a multiple of the 1-st one
" column " " " " " " ")

Example:

A rank-one $H(3,5)$ over $\mathbb{Z}_{31}$

$$\begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 5 & 10 & 20 & 9 & 18 \\ 25 & 19 & 7 & 14 & 28 \end{bmatrix}$$

IT exists iff the "equation"

$$L = X \cdot Y$$ is solvable

half-set of $\mathbb{F}_q$

zero-sum m-subset of $\mathbb{F}_q^*$

zero-sum n-subset of $\mathbb{F}_q^*$

COROLLARY IT certainly exists if $\boxed{q \equiv 3 \pmod 4}$ and $\boxed{\gcd(m, n) = 1}$

Proof $\mathbb{F}_q^{\square} = R_m \cdot R_n$

m-th roots of unity

n-th roots of unity

Work in progress:

Apart from Heffter arrays over $\mathbb{Z}_p$ with p a PUTATIVE Fermat prime $> 2^{2^{11}}+1$, "my" problem remains at the moment (July 2023) open only for pairs (m, n) of the form

$(p, 2p)$
(p, p^2)
$(p, 2p^2)$
$(2p, p^2)$

with p a (large) prime

It is natural (and useful) to extend the definition of a Heffter array as follows

DEFINITION (work in progress with Anita Pasotti)

A $\boxed{\text{Heffter Space}}$ is a G-additive resolvable partial linear space whose point-set is a half-set of G

We consider, in particular:

(v_r, b_k) Heffter Configurations

and

(r, n) Heffter Nets

N.B.: it is challenging to find them with a large r

Example of a $(20_3, 15_4)$-HC over $\mathbb{Z}_{41}$:

Points:

1, 2, 3, 4, 5, 6, 7, −8, 9, 10, 11, −12, 13, −14, 15, −16, −17, −18, 19, −20

Parallel classes:

1	3	4	−8
2	5	13	−20
6	−12	−17	−18
7	9	10	15
11	−14	−16	19

1	2	9	−12
3	6	13	19
4	5	7	−16
−8	11	15	−18
10	−14	−17	−20

1	6	7	−14
2	4	11	−17
3	5	10	−18
−8	9	19	−20
−12	13	15	−16

We have:

- An **HS** with r parallel classes $\forall\, r > 0$
- Recursive constructions which, starting from a single example, produce infinitely many HCs with greater v's but the same r
- A $(3, m)$-**HN** $\forall\, m \equiv 0 \pmod 4$ TOOL:  [ANITA FA LA FATINA]
- A $(4, m)$-**HN** $\forall$ odd m s.t. $2m^2+1$ is a prime power

TOOL: rank-one Heffter Arrays!!

- A $(9, 11)$-**HN** over $\mathbb{Z}_3^5$

The troubled birth of my favorite met

$q = 2m^2+1$ prime power with m odd.

A is a rank-one $H(m,m)$ with 1st column the group $X = \{x^0, x^1, \dots, x^{m-1}\}$ of m-th roots of unity. Hence the 1st row

$Y = \{y_0, y_1, \dots, y_{m-1}\}$ is a ZERO-SUM m-subset of $\mathbb{F}_q$.

Assume that $S \subset U(\mathbb{Z}_m)$ and that

$$x^0 y_0 + x^1 y_s + x^2 y_{2s} + \dots + x^{m-1} y_{(m-1)s} \quad \forall s \in S$$

Then A is "S-pantransversal"

(all transversals of slope $s \in S$ are ZERO-SUM)

BUT rows + columns + S-transversals **IS NOT** a PLS ☹

... unless $\gcd(m!, |S|) = 1$

Thus $|S| \geq 3 \Longrightarrow 3 \nmid m \Longrightarrow 3 \mid q \Longrightarrow \boxed{q = 3^5 \text{ and } m = 11}$

A $\{1,-2,-1\}$-pantransversal $H(9,9)$ over $\mathbb{Z}_{163}$

$$\begin{bmatrix}
1 & 2 & 160 & 142 & 119 & 84 & 36 & 128 & 143 \\
40 & 80 & 43 & 138 & 33 & 100 & 136 & 67 & 15 \\
133 & 103 & 90 & 141 & 16 & 88 & 61 & 72 & 111 \\
104 & 45 & 14 & 98 & 151 & 97 & 158 & 109 & 39 \\
85 & 7 & 71 & 8 & 9 & 131 & 126 & 122 & 93 \\
140 & 117 & 69 & 157 & 34 & 24 & 150 & 153 & 134 \\
58 & 116 & 152 & 86 & 56 & 145 & 132 & 89 & 144 \\
38 & 76 & 49 & 17 & 121 & 95 & 64 & 137 & 55 \\
53 & 106 & 4 & 28 & 113 & 51 & 115 & 101 & 81
\end{bmatrix}$$

(9, 11) Heffter Net over $\mathbb{F}_{3^5} = \mathbb{Z}_3[x]/(x^5+2x^4+1)$

Take the matrix

$$A = \begin{bmatrix}
0 & 1 & 18 & 3 & 81 & 27 & 54 & 162 & 6 & 9 & 2 \\
22 & 23 & 40 & 25 & 103 & 49 & 76 & 184 & 28 & 31 & 24 \\
44 & 45 & 62 & 47 & 125 & 71 & 98 & 206 & 50 & 53 & 46 \\
66 & 67 & 84 & 69 & 147 & 93 & 120 & 228 & 72 & 75 & 68 \\
88 & 89 & 106 & 91 & 169 & 115 & 142 & 8 & 94 & 97 & 90 \\
110 & 111 & 128 & 113 & 191 & 137 & 164 & 30 & 116 & 119 & 112 \\
132 & 133 & 150 & 135 & 213 & 159 & 186 & 52 & 138 & 141 & 134 \\
154 & 155 & 172 & 157 & 235 & 181 & 208 & 74 & 160 & 163 & 156 \\
176 & 177 & 194 & 179 & 15 & 203 & 230 & 96 & 182 & 185 & 178 \\
198 & 199 & 216 & 201 & 37 & 225 & 10 & 118 & 204 & 207 & 200 \\
220 & 221 & 238 & 223 & 59 & 5 & 32 & 140 & 226 & 229 & 222
\end{bmatrix}$$

replace each entry a_{ij} with $x^{a_{ij}}$ and then represent it as an element of $\mathbb{Z}_3^5$

$$
\begin{bmatrix}
10000 & 01000 & 11010 & 00010 & 11001 & 12102 & 02200 & 11022 & 22001 & 22220 & 00100 \\
12200 & 01220 & 20101 & 10011 & 01122 & 20100 & 02021 & 12110 & 11212 & 10110 & 00122 \\
11221 & 21120 & 12221 & 10210 & 00002 & 21222 & 10101 & 21110 & 02010 & 22000 & 02112 \\
21022 & 12101 & 22212 & 22122 & 12001 & 21221 & 00021 & 02120 & 12121 & 20210 & 21211 \\
21010 & 02101 & 12100 & 22022 & 02012 & 00222 & 11021 & 22221 & 22120 & 01022 & 20211 \\
02001 & 20201 & 11102 & 22200 & 21122 & 02201 & 21111 & 01112 & 10021 & 00210 & 22021 \\
20112 & 12010 & 22210 & 20121 & 21102 & 20020 & 20122 & 20021 & 20221 & 10222 & 01201 \\
10201 & 21021 & 00102 & 02210 & 01110 & 11122 & 20212 & 02121 & 02002 & 11101 & 22100 \\
00110 & 00011 & 12120 & 12002 & 11020 & 12202 & 10020 & 10220 & 11111 & 01211 & 20002 \\
10100 & 01010 & 01121 & 20011 & 00112 & 20222 & 02222 & 02100 & 11222 & 02111 & 00101 \\
12022 & 11201 & 12000 & 22110 & 22101 & 20001 & 01011 & 02202 & 12021 & 00212 & 21121
\end{bmatrix}
$$

The $(9, 11)$-Heffter Net is served!

It is $\{1, 2, 3, 5, 7, 9, 10\}$-pantransversal