Burning Steiner Triple Systems

Andrea Burgess, University of New Brunswick

Joint work with Caleb Jones and David Pike

Acknowledgements to Peter Danziger and Trent Marbach

Graph Burning

Graph burning was introduced by Bonato, Janssen and Roshanbin (2014) as a model for influence spread in a network.

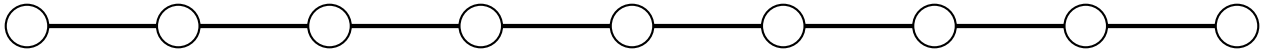
Given a graph G, in each round, the following happen simultaneously:

- An arsonist chooses a vertex to set on fire (a source).
- Existing fires spread to neigbouring vertices (propagation).

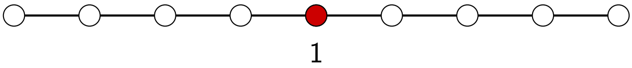
The process ends when every vertex is on fire.

The burning number $b(G)$ is the minimum number of rounds required to burn the entire graph.

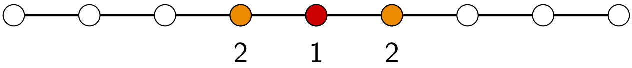
Example: Burning P_9



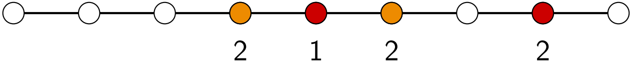
Example: Burning P_9



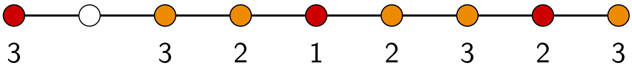
Example: Burning P_9



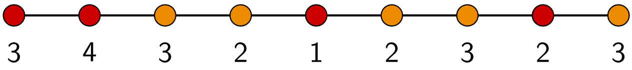
Example: Burning P_9



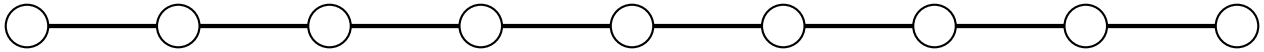
Example: Burning P_9



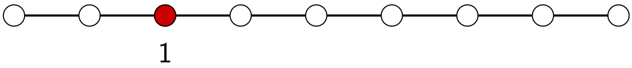
Example: Burning P_9



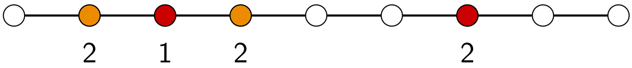
Example: Burning P_9



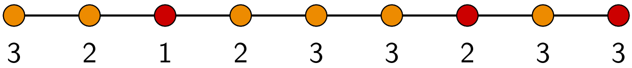
Example: Burning P_9



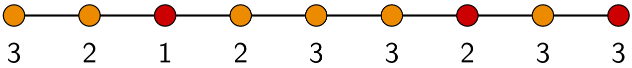
Example: Burning P_9



Example: Burning P_9



Example: Burning P_9



Theorem (Bonato, Janssen and Roshanbin, 2016)

$b(P_n) = \lceil \sqrt{n} \rceil$

Burning Number Conjecture

Conjecture (Bonato, Janssen and Roshanbin, 2016)

For a connected graph G of order n, $b(G) \leq \lceil \sqrt{n} \rceil$.

Burning Number Conjecture

Conjecture (Bonato, Janssen and Roshanbin, 2016)

For a connected graph G of order n, $b(G) \leq \lceil \sqrt{n} \rceil$.

Theorem (Kamali, Miller and Zhang, 2020)

The burning number conjecture holds for graphs of minimum degree at least 23.

Theorem (Bastide, Bonamy, Bonato, Charbit, Kamali, Pierron, Rabie, preprint)

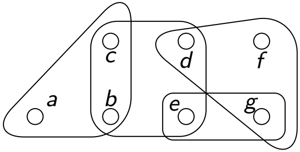
For a connected graph G of order n, $b(G) \leq 1 + \left\lceil \sqrt{\frac{4n}{3}} \right\rceil$.

Theorem (Norin and Turcotte, preprint)

For a connected graph G of order n, $b(G) \leq (1 + o(1))\sqrt{n}$.

Hypergraph Burning

A hypergraph is a pair $(V, \mathcal{E})$, where V is a finite set of vertices and $\mathcal{E} \subseteq \mathscr{P}(V)$. Elements of $\mathcal{E}$ are hyperedges or edges.



$$V = \{a, b, c, d, e, f, g\}$$

$$\mathcal{E} = \{\{a, b, c\}, \{b, c, d, e\}, \{d, f, g\}, \{e, g\}\}$$

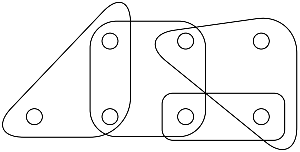
We would like a rule for burning hypergraphs that reduces to graph burning when all edges have size 2.

Hypergraph burning

In each round, the following occur simultaneously:

- Source: The arsonist selects a vertex to set on fire.
- Propagation: In any edge E, if all but one vertex of E is on fire, then the fire spreads to the remaining vertex of E.

The burning number $b(H)$ of a hypergraph H is the minimum number of rounds required to burn every vertex.

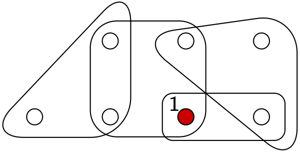


Hypergraph burning

In each round, the following occur simultaneously:

- Source: The arsonist selects a vertex to set on fire.
- Propagation: In any edge E, if all but one vertex of E is on fire, then the fire spreads to the remaining vertex of E.

The burning number $b(H)$ of a hypergraph H is the minimum number of rounds required to burn every vertex.

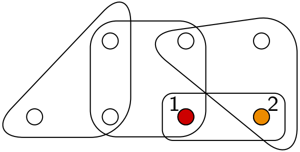


Hypergraph burning

In each round, the following occur simultaneously:

- Source: The arsonist selects a vertex to set on fire.
- Propagation: In any edge E, if all but one vertex of E is on fire, then the fire spreads to the remaining vertex of E.

The burning number $b(H)$ of a hypergraph H is the minimum number of rounds required to burn every vertex.

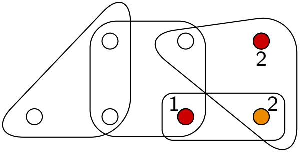


Hypergraph burning

In each round, the following occur simultaneously:

- Source: The arsonist selects a vertex to set on fire.
- Propagation: In any edge E, if all but one vertex of E is on fire, then the fire spreads to the remaining vertex of E.

The burning number $b(H)$ of a hypergraph H is the minimum number of rounds required to burn every vertex.

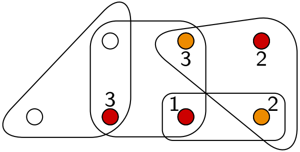


Hypergraph burning

In each round, the following occur simultaneously:

- Source: The arsonist selects a vertex to set on fire.
- Propagation: In any edge E, if all but one vertex of E is on fire, then the fire spreads to the remaining vertex of E.

The burning number $b(H)$ of a hypergraph H is the minimum number of rounds required to burn every vertex.

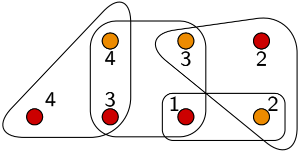


Hypergraph burning

In each round, the following occur simultaneously:

- Source: The arsonist selects a vertex to set on fire.
- Propagation: In any edge E, if all but one vertex of E is on fire, then the fire spreads to the remaining vertex of E.

The burning number $b(H)$ of a hypergraph H is the minimum number of rounds required to burn every vertex.

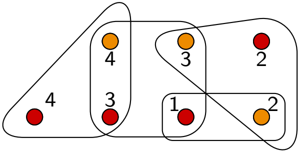


Hypergraph burning

In each round, the following occur simultaneously:

- Source: The arsonist selects a vertex to set on fire.
- Propagation: In any edge E, if all but one vertex of E is on fire, then the fire spreads to the remaining vertex of E.

The burning number $b(H)$ of a hypergraph H is the minimum number of rounds required to burn every vertex.



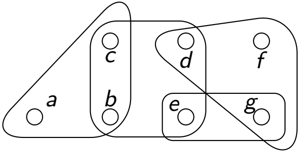
$b(H) = 4$

Lazy Burning

In lazy burning, the arsonist only gets one turn to set vertices on fire, but can burn multiple vertices in that turn.

Fire then propagates as before.

The lazy burning number $b_L(H)$ is the minimum number of vertices that the arsonist must set on fire so that all vertices become burnt.

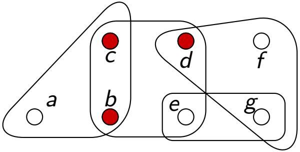


Lazy Burning

In lazy burning, the arsonist only gets one turn to set vertices on fire, but can burn multiple vertices in that turn.

Fire then propagates as before.

The lazy burning number $b_L(H)$ is the minimum number of vertices that the arsonist must set on fire so that all vertices become burnt.

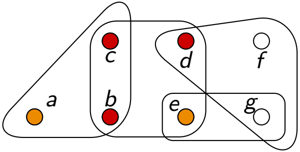


Lazy Burning

In lazy burning, the arsonist only gets one turn to set vertices on fire, but can burn multiple vertices in that turn.

Fire then propagates as before.

The lazy burning number $b_L(H)$ is the minimum number of vertices that the arsonist must set on fire so that all vertices become burnt.

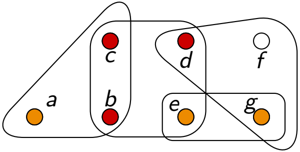


Lazy Burning

In lazy burning, the arsonist only gets one turn to set vertices on fire, but can burn multiple vertices in that turn.

Fire then propagates as before.

The lazy burning number $b_L(H)$ is the minimum number of vertices that the arsonist must set on fire so that all vertices become burnt.

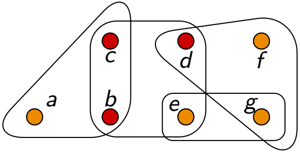


Lazy Burning

In lazy burning, the arsonist only gets one turn to set vertices on fire, but can burn multiple vertices in that turn.

Fire then propagates as before.

The lazy burning number $b_L(H)$ is the minimum number of vertices that the arsonist must set on fire so that all vertices become burnt.

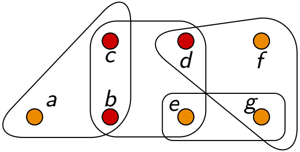


Lazy Burning

In lazy burning, the arsonist only gets one turn to set vertices on fire, but can burn multiple vertices in that turn.

Fire then propagates as before.

The lazy burning number $b_L(H)$ is the minimum number of vertices that the arsonist must set on fire so that all vertices become burnt.



$$b_L(H) = 3$$

A Few Bounds

Theorem (B., Jones and Pike, 2023+)

Let $H = (V, \mathcal{E})$ be a hypergraph with no isolated vertices, singleton edges or duplicate edges. Then

$$|V| - |\mathcal{E}| \leq b_L(H) < b(H) \leq \alpha(H) + 1,$$

where $\alpha(H)$ is the independence number of H, i.e. the maximum number of vertices that do not induce an edge.

A Few Bounds

Theorem (B., Jones and Pike, 2023+)

Let $H = (V, \mathcal{E})$ be a hypergraph with no isolated vertices, singleton edges or duplicate edges. Then

$$|V| - |\mathcal{E}| \leq b_L(H) < b(H) \leq \alpha(H) + 1,$$

where $\alpha(H)$ is the independence number of H, i.e. the maximum number of vertices that do not induce an edge.

Theorem (B., Jones and Pike, 2023+)

For any $k \in \mathbb{N}$, there exist hypergraphs G and H such that

$$b(G) - b_L(G) > k \text{ and } \frac{b(H)}{b_L(H)} > k.$$

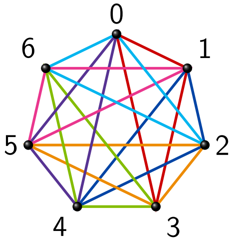
Steiner Triple Systems

Definition

A Steiner Triple system of order v, STS(v), is a pair $(V, \mathcal{E})$, where:

- $|V| = v$
- $\mathcal{E}$ is a collection of 3-subsets of V such that every pair of elements of V is contained in exactly one $E \in \mathcal{E}$.

That is, an STS(v) is a 3-uniform hypergraph of order v such that each pair of vertices is in exactly one edge.



$\{0, 1, 3\}$
$\{1, 2, 4\}$
$\{2, 3, 5\}$
$\{3, 4, 6\}$
$\{4, 5, 0\}$
$\{5, 6, 1\}$
$\{6, 0, 2\}$

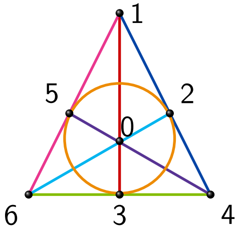
Steiner Triple Systems

Definition

A Steiner Triple system of order v, STS(v), is a pair $(V, \mathcal{E})$, where:

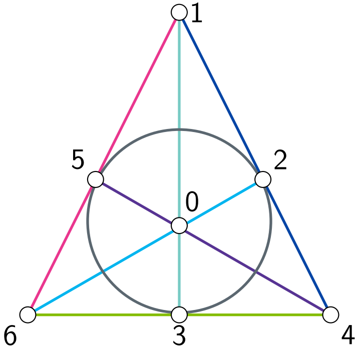
- $|V| = v$
- $\mathcal{E}$ is a collection of 3-subsets of V such that every pair of elements of V is contained in exactly one $E \in \mathcal{E}$.

That is, an STS(v) is a 3-uniform hypergraph of order v such that each pair of vertices is in exactly one edge.

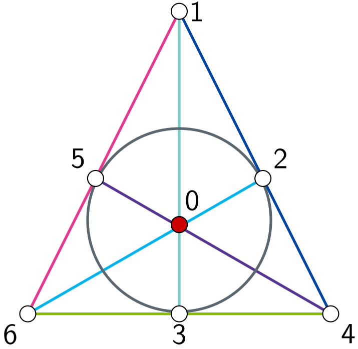


$\{0, 1, 3\}$
$\{1, 2, 4\}$
$\{2, 3, 5\}$
$\{3, 4, 6\}$
$\{4, 5, 0\}$
$\{5, 6, 1\}$
$\{6, 0, 2\}$

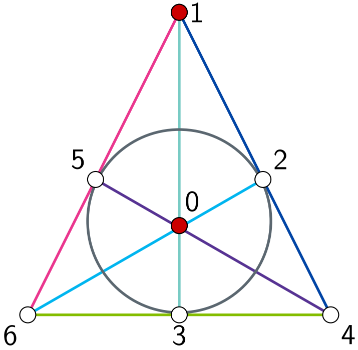
Example: Burning the STS(7)



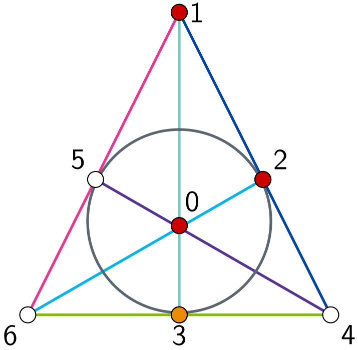
Example: Burning the STS(7)



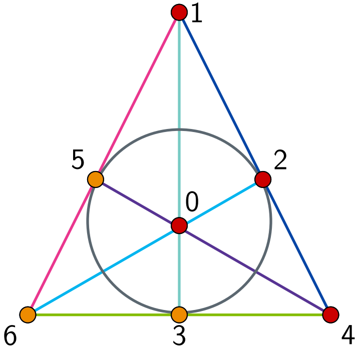
Example: Burning the STS(7)



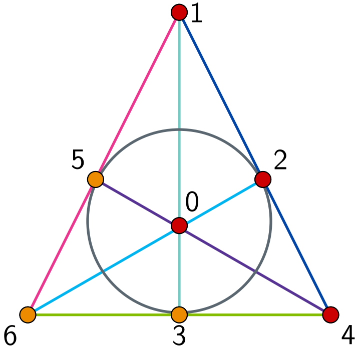
Example: Burning the STS(7)



Example: Burning the STS(7)

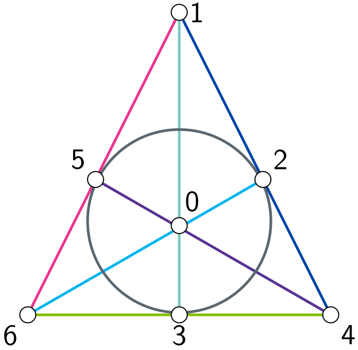


Example: Burning the STS(7)

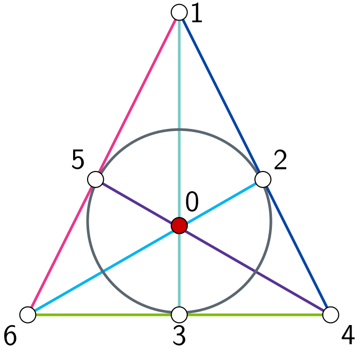


$b(H) = 4$

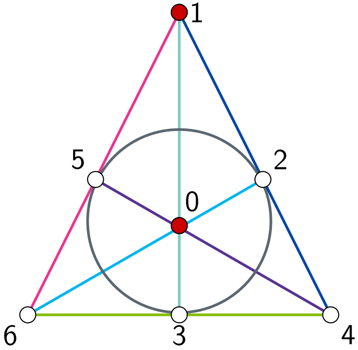
Example: Burning the STS(7)



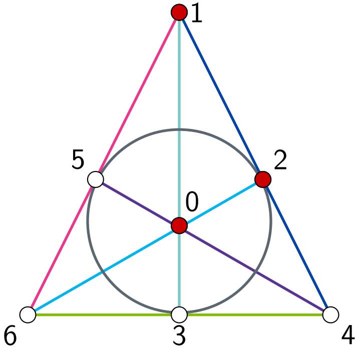
Example: Burning the STS(7)



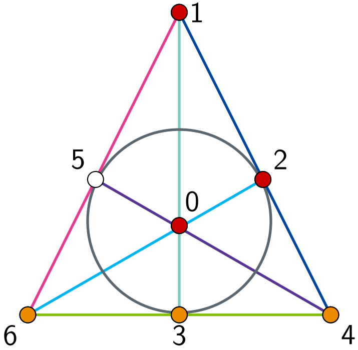
Example: Burning the STS(7)



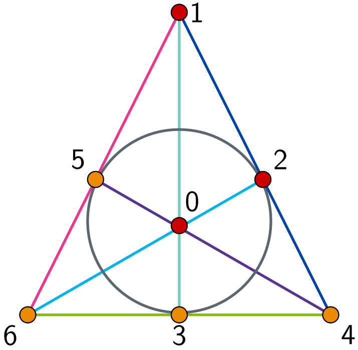
Example: Burning the STS(7)



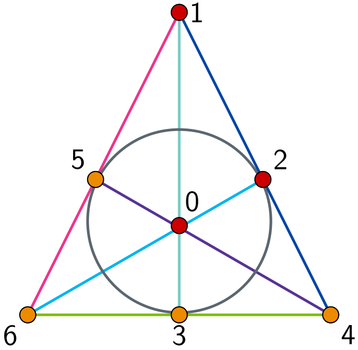
Example: Burning the STS(7)



Example: Burning the STS(7)



Example: Burning the STS(7)



$b_L(H) = 3$

A counting bound

Lemma (B., Jones and Pike, 2023+)

Let H be an $\mathrm{STS}(v)$. The maximum number of vertices burning after r rounds of the burning game is at most $h(r)$, where $h(1) = 1$ and for $r > 1$,

$$h(r) = \binom{h(r-1)}{2} - 2h(r-1) + 3r - 2.$$

Hence the least $r \in \mathbb{N}$ such that $v \leq h(r)$ is a lower bound on $b(H)$.

r	1	2	3	4	5	6	7	8
$h(r)$	1	2	4	8	25	266	34732	603069104

A counting bound

Lemma (B., Jones and Pike, 2023+)

Let H be an $\mathrm{STS}(v)$. The maximum number of vertices burning after r rounds of the burning game is at most $h(r)$, where $h(1) = 1$ and for $r > 1$,

$$h(r) = \binom{h(r-1)}{2} - 2h(r-1) + 3r - 2.$$

Hence the least $r \in \mathbb{N}$ such that $v \leq h(r)$ is a lower bound on $b(H)$.

r	1	2	3	4	5	6	7	8
$h(r)$	1	2	4	8	25	266	34732	603069104

Theorem (B., Jones and Pike, 2023+)

The burning number of a Steiner triple system can be arbitrarily large.

Constructing a lazy burning set

Consider the following procedure to construct a lazy burning set S:

1. Add an unburnt vertex to S.
2. Allow fire to propagate until no more vertices burn.

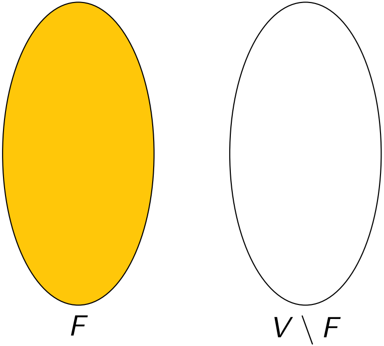
If these two steps are repeated until all vertices are on fire, then S is a lazy burning set.

What happens if we apply this procedure to a Steiner triple system?

Subsystems

Lemma (B., Jones and Pike, 2023+)

Let $H = (V, \mathcal{E})$ be an $\mathrm{STS}(v)$. Suppose that an arsonist sets fire to $S \subseteq V$, and that fire propagates until no new vertices burn. Then the burning vertices induce a sub-STS of H.



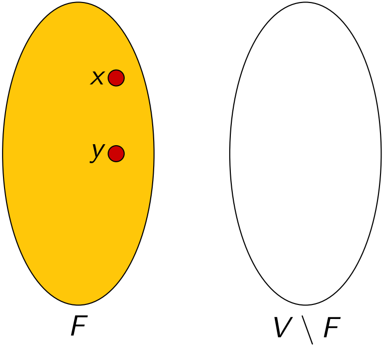
Let F be the set of burnt vertices after fire propagates.

For any $x, y \in F$, the third vertex of the triple containing x and y must be in F.

Subsystems

Lemma (B., Jones and Pike, 2023+)

Let $H = (V, \mathcal{E})$ be an $\mathrm{STS}(v)$. Suppose that an arsonist sets fire to $S \subseteq V$, and that fire propagates until no new vertices burn. Then the burning vertices induce a sub-STS of H.



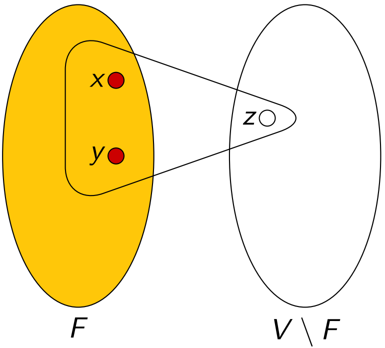
Let F be the set of burnt vertices after fire propagates.

For any $x, y \in F$, the third vertex of the triple containing x and y must be in F.

Subsystems

Lemma (B., Jones and Pike, 2023+)

Let $H = (V, \mathcal{E})$ be an $\mathrm{STS}(v)$. Suppose that an arsonist sets fire to $S \subseteq V$, and that fire propagates until no new vertices burn. Then the burning vertices induce a sub-STS of H.



Let F be the set of burnt vertices after fire propagates.

For any $x, y \in F$, the third vertex of the triple containing x and y must be in F.

Few subsystems

Theorem (Doyen, 1969)

For every admissible order v, there is an $\mathrm{STS}(v)$ with no nontrivial subsystems.

Theorem (B., Jones and Pike, 2023+)

For every admissible order v, there is an $\mathrm{STS}(v)$ with lazy burning number 3.

Many subsystems

Theorem (Folklore)

If X is an $\mathrm{STS}(v)$ and Y is an $\mathrm{STS}(w)$ which is a sub-STS of X, then $v \geq 2w + 1$.

Considering the worst case that there is a sub-system of each order $1, 3, 7, 15, \ldots$ we get:

Theorem (B., Jones and Pike, 2023+)

If H is an $\mathrm{STS}(v)$, then $b_L(H) \leq \lfloor \log_2(v + 1) \rfloor$.

Doubling Construction

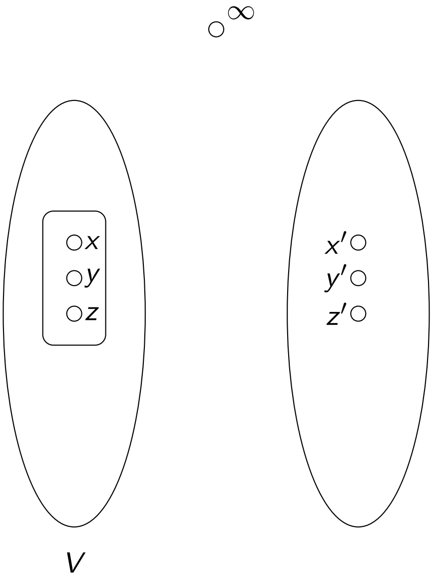
Let $H = (V, \mathcal{E})$ be an $\mathrm{STS}(v)$ on vertex set $\{1, 2, \ldots, v\}$. Define a $H^* = (V^*, \mathcal{E}^*)$ on vertex set $V \cup \{1', 2', \ldots, v'\} \cup \{\infty\}$ as follows:

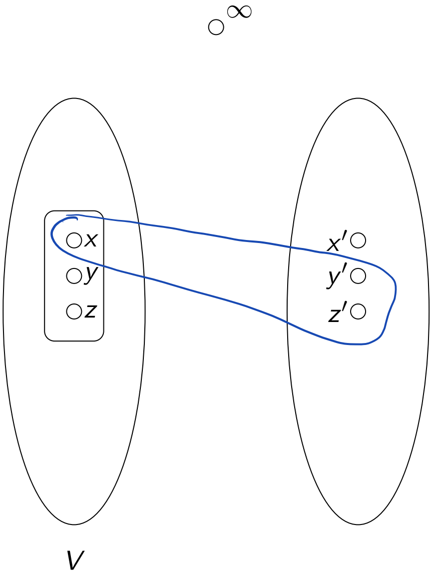
- For each $\{x, y, z\} \in \mathcal{E}$, the following are in $\mathcal{E}'$:

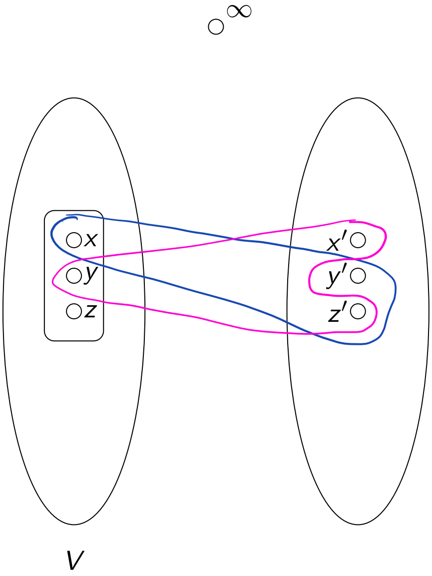
$$\{x, y, z\},\ \{x', y', z\},\ \{x', y, z'\},\ \{x, y', z'\}.$$

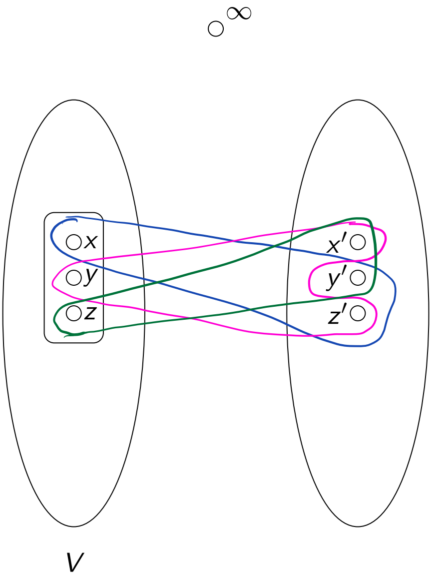
- For each $x \in V$, $\{x, x', \infty\} \in \mathcal{E}'$.

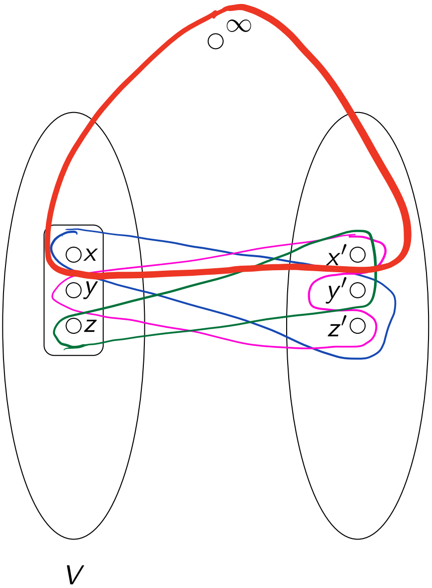
Then H^* is an $\mathrm{STS}(2v + 1)$ which contains H as a subsystem.


∞
x
y
z
x'
y'
z'
V


∞
x
y
z
x'
y'
z'
V


∞
x
y
z
x′
y′
z′
V


∞
x
y
z
x′
y′
z′
V



∞
x
y
z
x′
y′
z′
V

Theorem (B., Jones and Pike, 2023+)

If H is a Steiner triple system, then $b_L(H) \leq b_L(H^*) \leq b_L(H) + 1$.

Lemma (B., Jones and Pike, 2023+)

If H is a Steiner triple system, then $b_L(H^*) = b_L(H) + 1$ if and only if there is an optimal lazy burning set for H^* containing ∞.

Theorem (B., Jones and Pike, 2023+)

If H is a Steiner triple system, then $b_L(H) \leq b_L(H^*) \leq b_L(H) + 1$.

Lemma (B., Jones and Pike, 2023+)

If H is a Steiner triple system, then $b_L(H^*) = b_L(H) + 1$ *if and only if there is an optimal lazy burning set for* H^* *containing* ∞.

Theorem (B., Jones and Pike, 2023+)

If H is a cyclic Steiner triple system, then $b_L(H^*) = b_L(H) + 1$.

Idea.

If S is a lazy burning set that does not contain ∞, use an automorphism that cyclically permutes the points to map S to a lazy burning set containing ∞. □

Projective triple systems

Definition

Let $v = 2^n - 1$, where $n \geq 3$, and $V = \mathbb{Z}_2^n \setminus \{0\}$. The triples

$$\{\{x, y, z\} \subseteq V^3 \mid x + y + z = \mathbf{0}\}$$

form an $\mathrm{STS}(v)$, called the projective triple system of order n.

Theorem (B., Jones and Pike, 2023+)

If H is the projective triple system of order n, then $b_L(H) = n$.

Proof.

- The projective triple systems can be formed recursively by the doubling construction.
- Projective triple systems are cyclic (see, e.g. Singer, 1938).

□

Open problems / Future work

- Does applying the doubling construction to a cyclic STS yield a cyclic STS in general?
- Find an upper bound on the burning number of Steiner triple systems.
- What can we say about the burning number of BIBDs or other designs?
- Alternative propagation rules for burning hypergraphs.

Thanks!