Vertex-primitive s-arc-transitive digraphs of almost simple groups

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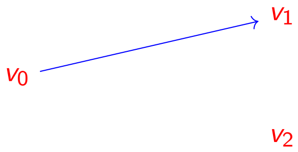

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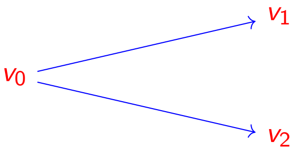

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Primitive Digraphs

A digraph Γ is called *G-vertex-primitive* if G acts primitively on the vertex set of Γ. Note that under this circumstance, G_v is maximal in G for any vertex $v \in \Gamma$.

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- $Cos(G, H, g)$ is $R_H(G)$-arc-transitive.

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Almost Simple Group

A group G is said to be *almost simple* if $Soc(G)$ is a non-abelian simple group.

Classification of Non-Abelian Simple Groups

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Conditions of Digraph

Self-paired Orbitals

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- $G = Sz(q), Ree(q)$, H is a maximal parabolic subgroup, then G acts 2-transitively on $Cos(G : H)$.

Cameron (1999)

For a group G acts on a set Ω, all of the orbitals of G are self-paired if and only if each constituent of the permutation character π is real and has multiplicity 1.

Cameron's Criteria

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Saxl (Unpublished)

For $G = Sp(2n, q)$, H a $\mathcal{C}_4$-maximal subgroup of G, then for the action of G on $Cos(G : H)$, the permutation character π is not multiplicity-free.

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- $W_1 \cap W_2^{\perp} = \langle e_4, f_4 \rangle$, $W_2 \cap W_1^{\perp} = \langle e_2, f_3 \rangle$.

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- $W_2 := \langle e_1, f_1, e_2, f_2, e_3, f_3, e_6, f_6 \rangle$.
- $W_1 \cap W_2^{\perp} = \langle e_4, f_4 \rangle$, $W_2 \cap W_1^{\perp} = \langle e_2, f_3 \rangle$.
- No element in $Sp_{12}(q)$ swaps W_1 and W_2.

Bounding s for Digraphs and Graphs

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Praeger(1989)

For all $s \geq 2$, there are infinitely many s-arc-transitive digraphs that are not $(s+1)$-arc-transitive..

Bounding s for Vertex-primitive Digraph

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Does there exist a vertex-primitive 2-arc-transitive digraph?

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It is sufficient to determine s when G is almost simple.

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Guess

All the vertex-primitive digraphs are at most 2-arc-transitive.

Progress so far

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- Pan-Wu-Yin(2020): $s \leq 2$ when $Soc(G) = A_n$ except for one subcase remains open.
- Chen-Giudici-Praeger(2023): $s \leq 1$ when $Soc(G) = Ree(q)$ or $Sz(q)$.
- Chen-Giudici-Prager(Unpublished): $s \leq 2$ when $(G) = PSp_{2n}(q), G_2(q), {}^3D_4(q), {}^2F_4(q)$.

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There exists an infinite family of G-vertex-primitive $(G, 2)$-arc-transitive digraph such that $Soc(G) = PSL_3(q)$ and $G_v \cap Soc(G) = A_6$.

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Question

If there exists any G-vertex-primitive $(G, 2)$-arc-transitive digraph of almost simple group such that G_v is of geometric type?