The Chromatic number
of some
generalized Kneser graphs

Joint work with Klaus Metsch and Daniel Werner

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GHENT UNIVERSITY

Overview

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2. Chromatic number line-plane flags in $PG(4, q)$
   - Examples of cocliques and colorings
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3. Chromatic number of $\{d-1, d\}$-flags in $PG(2d, q)$
   - Examples of cocliques and colorings
   - Results

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Notation.

- $\mathrm{PG}(n,q)$: the n-dimensional projective space over $\mathbb{F}_q$.
- $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \prod_{i=1}^{k+1} \frac{q^{n+1-i}-1}{q^i-1}$: the number of k-spaces in $\mathrm{PG}(n,q)$.
- $\theta_n = \begin{bmatrix} n+1 \\ 1 \end{bmatrix} = \frac{q^{n+1}-1}{q^1-1}$: the number of points in $\mathrm{PG}(n,q)$.

Definition.
A *flag* $\mathcal{F}$ is a set of subspaces in $\mathrm{PG}(n, q)$, s.t.
$\forall U, V \in \mathcal{F} : U \subsetneq V \vee V \subsetneq U$.

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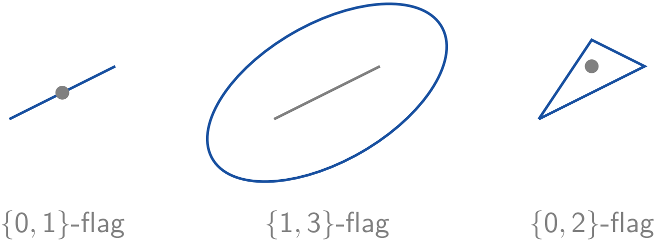
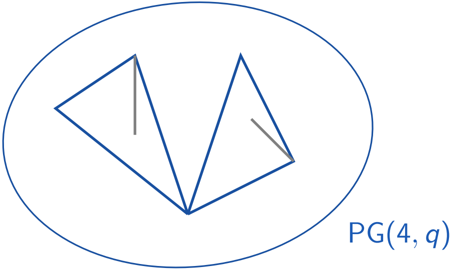


We always use projective dimensions.

Definition.
Two flags are in *general position* if
$\forall \pi_U \in U, \pi_V \in V : \pi_U \cap \pi_V = \emptyset \vee \langle \pi_U, \pi_V \rangle = \mathrm{PG}(n, q)$.



Two line-plane flags in PG(4, q) *in general position.*

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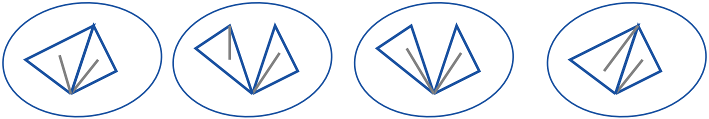

line-plane flags in $\mathrm{PG}(4, q)$*, not in general position.*

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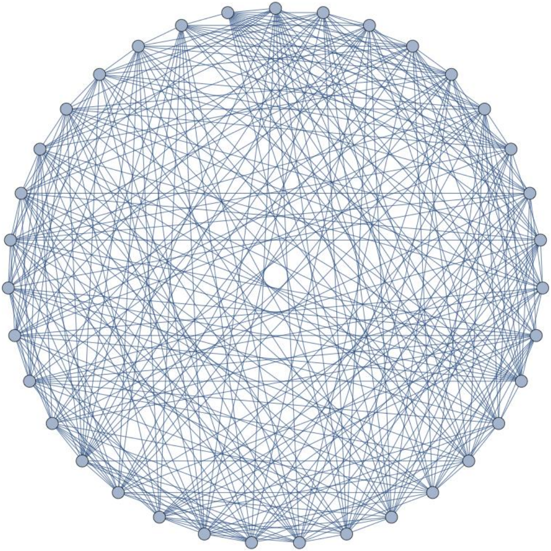
Definition.
The *q-Kneser graph* is the graph $qK_{n;\Omega}$, with vertices the flags of type Ω in $\mathrm{PG}(n, q)$ and two flags are adjacent if they are in general position.

Kneser graph $qK_{3;1}$, $q = 3$



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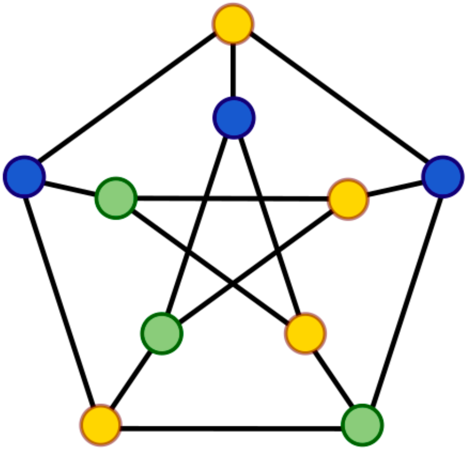
- For $\Gamma = qK_{n;\Omega}$, a coloring corresponds with a covering of all flags with EKR-sets of flags.



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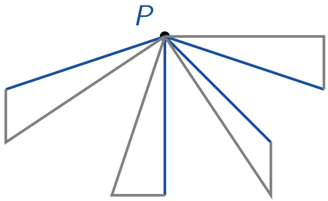
Point-pencil of line-plane flags in $\mathrm{PG}(4, q)$ through P,



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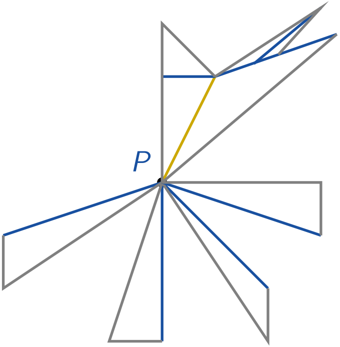
Point-based example

Point-pencil of line-plane flags in PG$(4, q)$ through P, together with a set of flags, whose planes pairwise intersect in a line through P. (Size $= q^5 + 3q^4 + 4q^3 + 4q^2 + 2q + 1$)



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Hyperplane-based example

The dual of a *point-based* example.

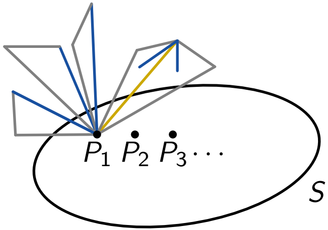
Theorem ([BB17]).

Every EKR-set of line-plane flags of $\mathrm{PG}(4, q)$, which is not a subset of one of the sets defined above, has cardinality at most

$$4q^4 + 9q^3 + 4q^2 + q + 1.$$

Example of a covering

1. All point-based examples with base point in a solid S.



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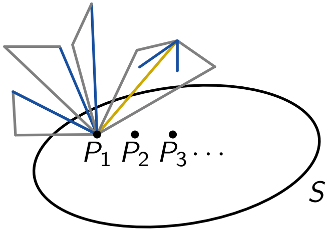

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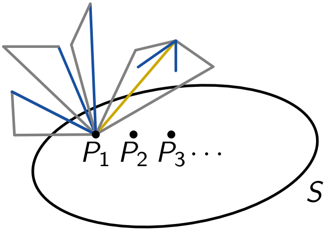

We know that $\chi(qK_{4;\{1,2\}}) \leq q^3 + q^2 + 1$.

Strategy

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4. Crucial lemma on point sets.

Lemma on point sets

Lemma.
Suppose that M is a set of points in $PG(4,q)$, and P_1, P_2, P_3 are three non-collinear points such that the plane π they span has no point in M. Let m, n and d be positive real numbers such that the following hold:

- Each of the points P_1, P_2, P_3 lies on at most nq^2 lines that meet M,
- $|M| = dq^3$,
- $q > 32n^5 m/d^5$.

Then there exists a solid S on π with $|S \cap M| \geq mq^2$.

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4. Crucial lemma on point sets, which defines the solid S.
5. Using counting arguments and the crucial lemma, we find that all elements of C are point-based examples with base point contained in a solid S, and $|C| = q^3 + q^2 + 1$.

Main result

Theorem.
For $q > 160 \cdot 36^5$ the chromatic number of the Kneser graph $qK_{4;\{1,2\}}$ is $q^3 + q^2 + 1$. Up to duality, each color class C of a minimum coloring is contained in a unique point-based example, and the base points of these point-based examples are $q^3 + q^2 + 1$ distinct points of a solid.

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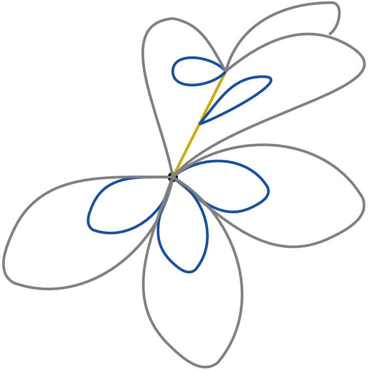

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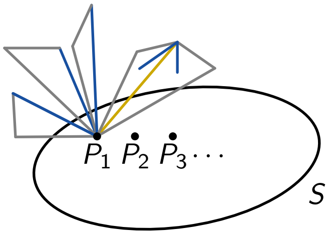
We need a stability result on large EKR-sets of flags.

- For $d = 2$ and $d = 3$ there is a result known.
- For $d > 3$ there is no result known yet.
  $\Rightarrow$ We use a conjecture.

Example of a covering

1. $q^{d+1} + q^d + \cdots + q^2 + 1$ point-based examples with base point in a $(d+1)$-space S. We use the special part of these sets to cover the remaining flags.



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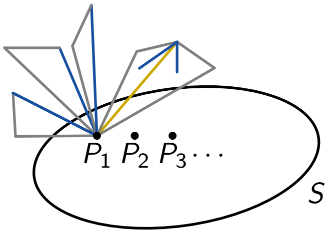

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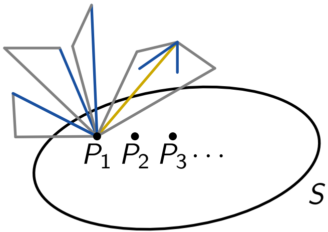

We know that $\chi(qK_{2d;\{d-1,d\}}) \leq q^{d+1} + q^d + \cdots + q^2 + 1$.

Main result

Conjecture.
For $d \geq 2$ there is an integer $\rho(d)$ such that every maximal coclique of $q\Gamma_{2d,\{d-1,d\}}$ contains a point-pencil, a dual point-pencil, or has at most $\rho(d) \cdot q^{d^2+d-2}$ elements.

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This conjecture is true for $d = 2$, see [BB17], and for $d = 3$, see [MW20]

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Theorem.
If the conjecture is true for some integer $d \geq 2$, then

$$\chi(q\Gamma_{2d,\{d-1,d\}}) = q^{d+1} + q^d + \cdots + q^2 + 1$$

for sufficiently large q. Moreover, if $\mathcal{F}$ is a family of this many maximal cocliques that cover the vertex set, then – up to duality – there exists a $(d+1)$-dimensional subspace U, such that all elements of $\mathcal{F}$ are contained in point-based examples, based on a point in U.

References

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Thank you very much for your attention.