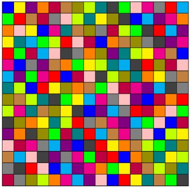
The Hadamard quasigroup product of orthogonal Latin squares



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The classical Hadamard product.

Jacques **Hadamard**

(1865-1963)

$$\boxed{(A \odot B)\,[i,j] := A[i,j] \cdot B[i,j]}$$

Hadamard product $\equiv$ Element-wise product.

Example:

$A \equiv$

1	−3	2
2	0	−1
−1	2	3

$B \equiv$

0	2	−1
−1	−3	2
−1	1	3

$\Rightarrow A \odot B \equiv$

0	−6	−2
−2	0	−2
1	2	9

n-ary quasigroups and Latin hypercubes.

Ruth **Moufang** (1935)

A **quasigroup** is a pair (X, f) formed by

- a set X, and
- a binary operation $f : X \times X \to X$,

such that both equations

$$f(a, x) = b \quad \text{and} \quad f(y, a) = b$$

have unique solutions $x, y \in X$, for all $a, b \in X$.

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- Its Cayley table is a **Latin square**.

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An ***n*-ary quasigroup** is a pair (X, f) formed by a set X and an n-ary operation $f : X^n \to X$ such that the equation

$$f(x_1, \ldots, x_n) = y$$

has unique solution in X, whenever $n - 1$ variables in $\{x_1, \ldots, x_n\}$, and also the variable y, are fixed.

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- Its Cayley table is a **Latin hypercube** of dimension n.

Generalizing the Hadamard product.

- $\Omega_n(X) := \{n\text{-ary operations on a set } X\}$
- $\overline{x} := (x_1, \ldots, x_n) \in X^n$.

$$\begin{array}{ccc} \Omega_m(X) & \to & \Omega_m(\Omega_n(X)) \\ f & \to & \odot_f : \begin{array}{ccc} (\Omega_n(X))^m & \to & \Omega_n(X) \\ (g_1, \ldots, g_m) & \to & \odot_f(g_1, \ldots, g_m) \end{array} \end{array}$$

$$\boxed{\odot_f(g_1, \ldots, g_m)(\overline{x}) := f(g_1(\overline{x}), \ldots, g_m(\overline{x}))}$$

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\end{array}
$$

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$m = n = 2$:

$$\boxed{\odot_f(g_1, g_2)(x, y) := f\left(g_1(x, y),\ g_2(x, y)\right)}$$

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In the usual notation for binary operations:

$$g_1 \equiv \triangle \qquad g_2 \equiv \square \qquad f \equiv \star$$

$$\boxed{\triangle \odot_\star \square \to x \triangle \odot_\star \square\, y := (x \triangle y) \star (x \square y)}$$

Generalizing the Hadamard product.

$$\triangle \equiv \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 0 & 2 & 2 \\ \hline \end{array} \qquad \square \equiv \begin{array}{|c|c|c|} \hline 0 & 2 & 1 \\ \hline 2 & 1 & 2 \\ \hline 1 & 1 & 0 \\ \hline \end{array} \qquad \star \equiv \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array}$$

$$\triangle \odot_{\star} \square \equiv \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 1 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline \end{array}$$

Generalizing the Hadamard product.

Theorem (Fuchs'58)

$$\mathrm{Mult}_f(X) := \{g \in \Omega_2(X) \colon\ (X, f, g) \text{ is a ring}\}\,.$$

1. *(X, f) abelian group $\Rightarrow$ $(\mathrm{Mult}_f(X), \odot_f)$ abelian group.*
2. $\mathrm{Mult}_f(X) \cong \mathrm{Hom}\,(X,\ \mathrm{End}(X))$.

Theorem (Clay'68)

$$\mathrm{Mult}_{f\,L}(X) := \{g \in \Omega_2(X) \colon\ (X, f, g) \text{ is a near-ring}\}\,.$$

1. *$(X, \star)$ abelian group $\Rightarrow$ $(\mathrm{Mult}_{f\,L}(X), \odot_f)$ abelian group.*
2. $\mathrm{Mult}_{f\,L}(X) \cong \mathrm{Map}\,(X,\ \mathrm{End}(X))$.

Generalizing the Hadamard product.

Lemma

(X, f) embeds into $(\Omega_n(X), \odot_f)$ via the homomomorphism

$$\begin{array}{ccc} (X, f) & \hookrightarrow & (\Omega_n(X), \odot_f) \\ c & \rightarrow & \begin{array}{ccc} X^n & \rightarrow & X \\ \overline{x} & \rightarrow & c \end{array} \end{array}$$

Generalizing the Hadamard product.

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Every algebraic identity satisfied by (X, f) is also satisfied by $(\Omega_n(X), \odot_f)$.

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Example($m = n = 2$) : Commutativity $(\triangle \odot_\star \Box \equiv \Box \odot_\star \triangle)$

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$$x\triangle \odot_\star \square y = (x\triangle y) \star (x\square y) = (x\square y) \star (x\triangle y) = x\square \odot_\star \triangle y$$

The Hadamard quasigroup product.

- $\mathcal{Q}_m(X) := \{f \in \Omega_m(X) \colon (X, f) \text{ is an } m\text{-ary quasigroup}\}$.

Proposition

$f \in \mathcal{Q}_m(X) \Leftrightarrow \odot_f \in \mathcal{Q}_m\left(\Omega_n(X)\right)$.

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Lemma ($m = n$)

Let $(g_1, \ldots, g_n) \in (\Omega_n(X))^n$. The map

$$\begin{array}{ccc} \Omega_n(X) & \to & \Omega_n(X) \\ f & \to & \odot_f(g_1, \ldots, g_n) \end{array}$$

is a permutation if and only iff the set $\{g_1, \ldots, g_n\}$ is orthogonal. That is, iff the map

$$\begin{array}{ccc} X^n & \to & X^n \\ \overline{x} & \to & (g_1(\overline{x}), \ldots, g_n(\overline{x})) \end{array}$$

is a permutation.

The Hadamard quasigroup product.

$\mathcal{O}(g_1,\ldots,g_n) := \{g \in \Omega_n : \ (\{g_1,\ldots,g_n\} \setminus \{g_i\}) \cup \{g\}$ orthogonal, for all $i\}$

Proposition

If $\{g_1,\ldots,g_n\}$ is orthogonal, then the map

$$\begin{array}{ccc} \mathcal{Q}_n(X) & \to & \mathcal{O}(g_1,\ldots,g_n) \\ f & \to & \odot_f(g_1,\ldots,g_n) \end{array}$$

is a bijection.

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is a bijection.

Under which conditions $\odot_f(g_1, \ldots, g_n) \in \mathcal{Q}_n(X)$?

$$(m = n = 2)$$

The Hadamard quasigroup product ($m = n = 2$).

The Hadamard quasigroup product does not preserve the Latin square property in general.

$$f \equiv \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 1 & 2 \\ \hline 2 & 3 & 1 \\ \hline \end{array} \Rightarrow f^2 := \odot_f(f, f) \equiv \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

Lemma

Let $f \in \mathcal{Q}_2(X)$ and let $g_1, g_2 \in \Omega_2(X)$. Then, $\odot_f(g_1, g_2) \in \mathcal{Q}_2(X)$ iff

$$\{(g_1(x, y), g_2(x, y)) : y \in X\}$$

and

$$\{(g_1(y, x), g_2(y, x)) : y \in X\}$$

are Latin transversals in (X, f), for all $x \in X$.

The Hadamard quasigroup product ($m = n = 2$).

Proposition

$$f^2 \in \mathcal{Q}_2(X) \Leftrightarrow X = \{f(x,x) \colon\ x \in X\}$$

$f \equiv f^2 \equiv$

1	3	2
3	2	1
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The Hadamard quasigroup product ($m = n = 2$).

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3	2	1
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What about successive iterations?

The Hadamard quasigroup product $(m = n = 2)$.

$$\odot_\ell^2 f := \odot_\rho^2 f := f^2.$$

$$\odot_\ell^k f := \odot_f \left(f,\, \odot_\ell^{k-1} f\right) \qquad \text{and} \qquad \odot_\rho^k f := \odot_f \left(\odot_\rho^{k-1} f,\, f\right)$$

Proposition

The minimum positive integers $\ell(f)$ and $\rho(f)$ such that

$$\odot_\ell^{\ell(f)+1} f = \odot_\rho^{\rho(f)+1} f = f$$

are quasigroup isomorphism invariants. They satisfy that

$$\ell(f) = \rho(f^t).$$

The Hadamard quasigroup product ($m = n = 2$).

$$\odot_\ell^2 f := \odot_\rho^2 f := f^2.$$

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2	4	1	3	5
1	3	5	2	4
5	2	4	1	3
4	1	3	5	2
3	5	2	4	1

$(\rho, \ell) = (3, 5)$

2	1	5	4	3
4	3	2	1	5
1	5	4	3	2
3	2	1	5	4
5	4	3	2	1

$(\rho, \ell) = (5, 3)$

2	5	3	1	4
5	3	1	4	2
3	1	4	2	5
1	4	2	5	3
4	2	5	3	1

$(\rho, \ell) = (5, 5)$

The Hadamard quasigroup product $(m = n = 2)$.

2	5	12	4	8	7	13	14	15	16	3	6	10	11	1	9
3	1	7	8	11	6	14	15	5	13	16	2	9	12	10	4
4	6	5	2	10	15	7	13	11	8	14	3	16	1	9	12
5	15	9	3	14	1	6	8	2	4	7	13	11	16	12	10
1	3	8	9	6	16	11	4	13	12	15	14	5	10	7	2
10	2	11	12	7	4	3	5	14	15	9	16	1	13	8	6
9	12	3	10	1	2	8	6	16	14	13	15	4	5	11	7
15	14	2	1	5	13	4	7	3	6	8	10	12	9	16	11
13	7	14	15	16	3	5	9	10	11	6	12	2	8	4	1
14	10	15	7	13	5	16	2	12	9	11	1	8	4	6	3
16	13	4	5	15	14	1	11	7	10	12	9	3	6	2	8
12	4	16	13	3	8	15	1	9	7	10	11	6	2	14	5
6	8	10	11	9	12	2	16	1	5	4	7	14	3	15	13
7	9	6	16	12	11	10	3	8	1	2	4	13	15	5	14
11	16	1	6	2	9	12	10	4	3	5	8	7	14	13	15
8	11	13	14	4	10	9	12	6	2	1	5	15	7	3	16

$$\rho = 30$$

$$\mathrm{MO}(k, X) := \left\{(f_1, \ldots, f_k) \in (\Omega_2(X))^k : \; f_i \perp f_j, \text{for all } i, j\right\}.$$

$$\mathrm{MO}_2(k, X) := \left\{(f_1, f_2, g_1, \ldots, g_{k-2}) \in \mathrm{MO}(k, X) : \; \begin{cases} f_1, f_2 \in \Omega_2(X), \\ g_1, \ldots, g_{k-2} \in \mathcal{Q}_2(X) \end{cases}\right\}.$$

Lemma

The following map is an involution.

$$\begin{array}{rccc} \Phi : & \mathrm{MO}_2(k, X) & \to & \mathrm{MO}_2(k, X) \\ & (f_1, f_2, g_1, \ldots, g_{k-2}) & \to & (f_1^\Phi, f_2^\Phi, g_1^\Phi, \ldots, g_{k-2}^\Phi) \end{array}$$

where

$$\begin{array}{rcl} f_1^\Phi\,(f_1(x, y),\ f_2(x, y)) & := & x \\ f_2^\Phi\,(f_1(x, y),\ f_2(x, y)) & := & y \\ g_s^\Phi\,(f_1(x, y),\ f_2(x, y)) & := & g_s(x, y), \end{array}$$

for all $s \in \{1, \ldots, k-2\}$.

		f_1	f_2	g_1	$\dots$	g_{k-2}
x	y	$f_1(x,y)$	$f_2(x,y)$	$g_1(x,y)$	$\dots$	$g_{k-2}(x,y)$

$$\updownarrow$$

f_1	f_2			g_1	$\dots$	g_{k-2}
$f_1(x,y)$	$f_2(x,y)$	x	y	$g_1(x,y)$	$\dots$	$g_{k-2}(x,y)$

$$\updownarrow$$

		f_1^Φ	f_2^Φ	g_1^Φ	$\dots$	g_{k-2}^Φ
$f_1(x,y)$	$f_2(x,y)$	x	y	$g_1(x,y)$	$\dots$	$g_{k-2}(x,y)$

		f_1	f_2	g_1	$\dots$	g_{k-2}
x	y	$f_1(x,y)$	$f_2(x,y)$	$g_1(x,y)$	$\dots$	$g_{k-2}(x,y)$

$$\updownarrow$$

f_1	f_2			g_1	$\dots$	g_{k-2}
$f_1(x,y)$	$f_2(x,y)$	x	y	$g_1(x,y)$	$\dots$	$g_{k-2}(x,y)$

$$\updownarrow$$

		f_1^Φ	f_2^Φ	g_1^Φ	$\dots$	g_{k-2}^Φ
$f_1(x,y)$	$f_2(x,y)$	x	y	$g_1(x,y)$	$\dots$	$g_{k-2}(x,y)$

If $f_1,\ f_2 \in \mathcal{Q}_2(X)$, then $(f_1, f_2, g_1, \dots g_{k-2})$ and $\left(f_1^\Phi, f_2^\Phi, g_1^\Phi, \dots, g_{k-2}^\Phi\right)$ are two *paratopic* k-MOLS [Egan, Wanless'16].

$$\begin{array}{rcl} f_1^{\Phi}\left(f_1(x,y),\ f_2(x,y)\right) & := & x \\ f_2^{\Phi}\left(f_1(x,y),\ f_2(x,y)\right) & := & y \\ g_s^{\Phi}\left(f_1(x,y),\ f_2(x,y)\right) & := & g_s(x,y), \end{array}$$

Lemma

For each $s \in \{1, \ldots, k-2\}$,

$$\odot_{g_s^{\Phi}}(f_1, f_2) = g_s$$

and

$$\odot_{g_s}\left(f_1^{\Phi},\ f_2^{\Phi}\right) = g_s^{\Phi}.$$

f_1	f_2	g_1	$\ldots$	g_{k-2}

$\updownarrow \Phi$

f_1^{Φ}	f_2^{Φ}	$\odot_{g_1}\left(f_1^{\Phi},\ f_2^{\Phi}\right)$	$\ldots$	$\odot_{g_s}\left(f_1^{\Phi},\ f_2^{\Phi}\right)$

The Hadamard quasigroup product of orthogonal binary operations.

$$\begin{aligned} f_1^{\Phi}\left(f_1(x,y),\ f_2(x,y)\right) &:= x \\ f_2^{\Phi}\left(f_1(x,y),\ f_2(x,y)\right) &:= y \\ g_s^{\Phi}\left(f_1(x,y),\ f_2(x,y)\right) &:= g_s(x,y), \end{aligned}$$

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For each $s \in \{1, \ldots, k-2\}$,

$$\odot_{g_s^{\Phi}}(f_1, f_2) = g_s$$

and

$$\odot_{g_s}\left(f_1^{\Phi},\ f_2^{\Phi}\right) = g_s^{\Phi}.$$

f_1	f_2	$g_1(f_1, f_2)$	$\ldots$	$g_{k-2}(f_1, f_2)$

$$\updownarrow \Phi$$

f_1^{Φ}	f_2^{Φ}	g_1	$\ldots$	g_{k-2}

Theorem

If $\{g_1, g_2\}$ is orthogonal, then the following map is a bijection.

$$\begin{array}{ccc} \mathcal{O}(g_1, g_2) \cap \mathcal{Q}_2(X) & \to & \mathcal{O}\left(g_1^{\Phi}, g_2^{\Phi}\right) \cap \mathcal{Q}(X) \\ f & \to & \odot_f \left(g_1^{\Phi}, g_2^{\Phi}\right) \end{array}$$

1	2	3	4	5	6	7	8
2	1	4	3	8	7	6	5
3	5	2	6	1	8	4	7
5	3	1	7	2	4	8	6
8	4	7	1	6	3	5	2
4	8	6	2	7	5	3	1
7	6	5	8	3	2	1	4
6	7	8	5	4	1	2	3

g_1

1	2	3	4	5	6	7	8
3	5	1	7	2	8	4	6
2	1	8	5	4	7	6	3
8	4	6	2	7	3	5	1
6	7	5	8	3	1	2	4
5	3	2	6	1	4	8	7
4	8	7	1	6	5	3	2
7	6	4	3	8	2	1	5

g_2

1	2	3	4	5	6	7	8
5	3	7	1	6	4	8	2
8	4	1	7	2	3	5	6
7	6	8	5	4	2	1	3
4	8	2	6	1	5	3	7
6	7	4	3	8	1	2	5
3	5	6	2	7	8	4	1
2	1	5	8	3	7	6	4

f

1	8	7	3	2	4	6	5
8	1	2	5	7	6	4	3
5	3	1	4	8	7	2	6
2	7	4	1	6	3	5	8
3	5	8	6	1	2	7	4
4	6	5	2	3	1	8	7
6	4	3	7	5	8	1	2
7	2	6	8	4	5	3	1

g_1^{Φ}

1	6	7	5	2	3	8	4
7	2	1	8	6	4	5	3
6	1	3	2	8	5	4	7
3	8	6	4	1	7	2	5
2	7	4	6	5	8	3	1
8	3	5	7	4	6	1	2
5	4	8	1	3	2	7	6
4	5	2	3	7	1	6	8

g_2^{Φ}

1	7	4	2	3	8	5	6
6	2	5	7	8	3	4	1
5	8	3	6	4	7	1	2
7	1	2	4	6	5	8	3
4	3	8	1	5	2	6	7
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8	5	6	3	2	1	7	4
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$\odot_{f^{\Phi}} \left(g_1^{\Phi}, g_2^{\Phi}\right)$

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