On the trivial T-module

Rijeka conference on combinatorial objects and their applications

July 3-7, 2023, University of Rijeka
Rijeka, Croatia

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UNIVERSITÀ DEL LITORALE
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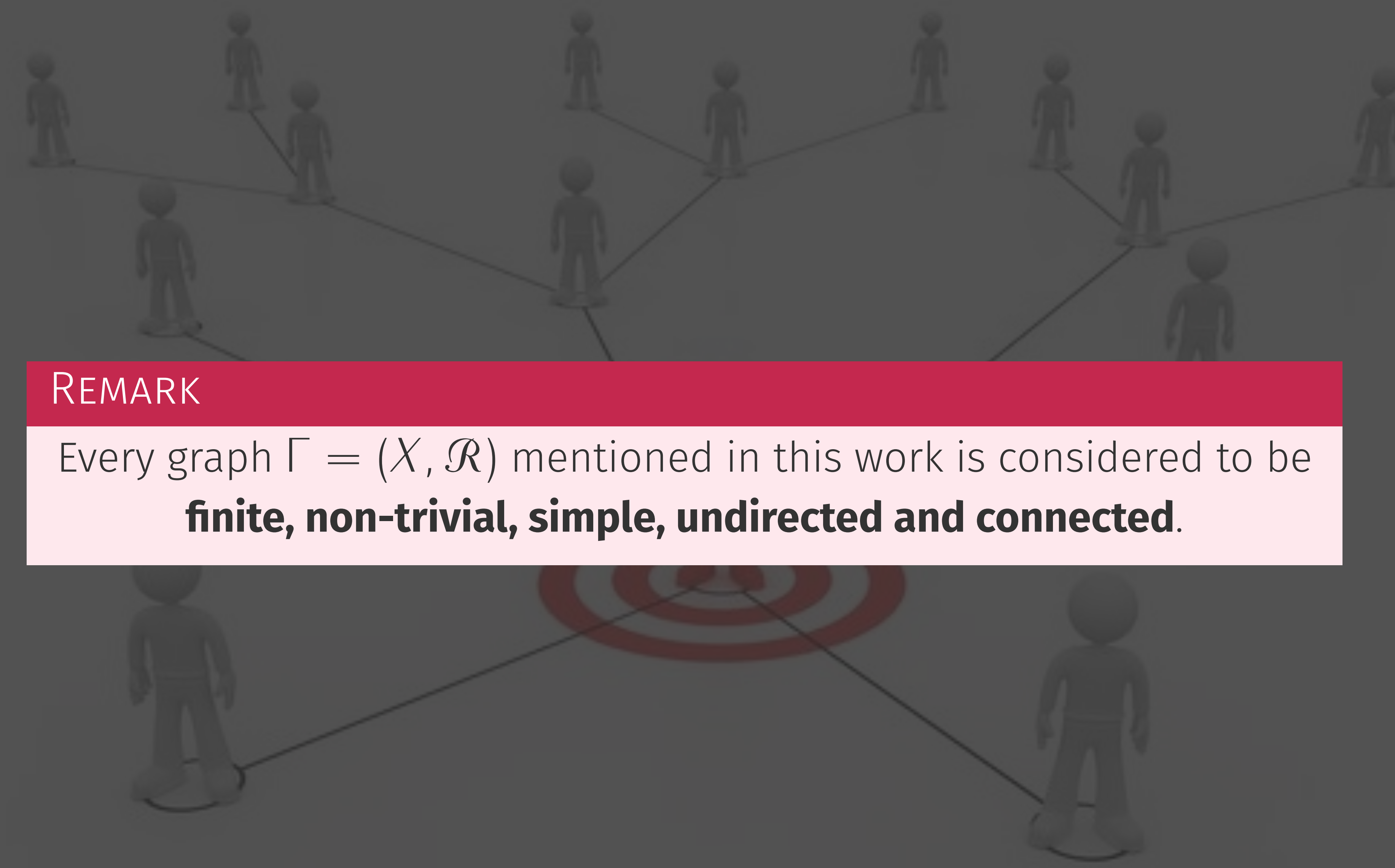


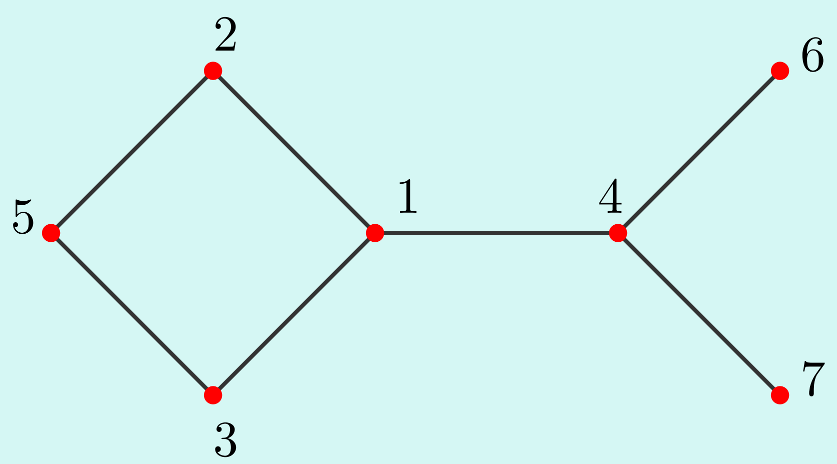
Remark

Every graph $\Gamma = (X, \mathscr{R})$ mentioned in this work is considered to be **finite, non-trivial, simple, undirected and connected**.

Example

Let $\Gamma = (X, \mathcal{R})$ be the simple graph with $X = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$\mathcal{R} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,5\}, \{3,5\}, \{4,6\}, \{4,7\}\}.$$



Motivation

Main Motivation

Let Γ be a graph and let G denote a certain algebraic object.

Question

What could we say about the combinatorial properties of Γ, if we know that G has certain algebraic properties?

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Question

What could we say about the algebraic properties of G, if we know that Γ has certain combinatorial properties?

Terwilliger algebra of a graph

Kranjska Gora
June 2023

Preliminaries

Terminology and Notations

Let $\Gamma = (X, \mathcal{R})$ be a graph and let $x, y \in X$.

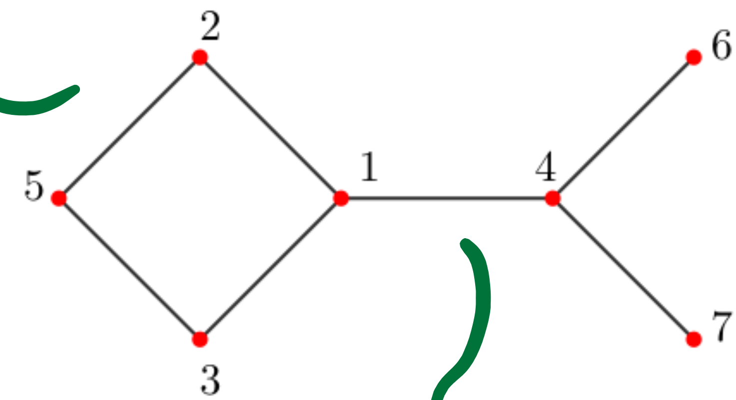
- The **distance** between x and y, denoted $\partial(x, y)$, is the length of a shortest xy-path.
- The **eccentricity of** x is the greatest distance between x and any other vertex. That is, $\epsilon(x) := \max_{z \in X} \partial(x, z)$.
- For $i \in \mathbb{Z}$, the collection of all vertices which are at distance i from vertex x is represented by $\Gamma_i(x)$. We abbreviate $\Gamma(x) = \Gamma_1(x)$.
- The collection of all the subsets $\Gamma_i(x)$, for $0 \leq i \leq \epsilon(x)$, makes up a partition of the vertex set X which is called the **distance partition of** Γ **relative to** x.

Terminology and Notations

Let $\Gamma = (X, \mathscr{R})$ be a graph and let $\mathbb{C}$ denote the complex number field.

- The $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$ is denoted by $\mathrm{Mat}_X(\mathbb{C})$
- The **standard module**, indicated by $V = \mathbb{C}^X$, is the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$.
- The vector space V is endowed with the Hermitian inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies $\langle u, v \rangle = u^t \overline{v}$ for $u, v \in V$, where t means transpose and $\overline{}$ symbolises complex conjugation.
- For $y \in X$, let $\widehat{y} \in V$ be the vector with a 1 in the y-coordinate and zeros everywhere else.
- Note that the set $\{\hat{y} : y \in X\}$ is an orthonormal basis for V.

$\mathbb{C}$-ALGEBRA OF ALL 7 × 7 MATRICES



$$Mat_X(\mathbb{C}) = Mat_7(\mathbb{C}) = \mathbb{C}^{7\times 7}$$

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array}
\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix} \end{array}$$

COORDINATES INDEXED BY THE VERTEX SET (IN THE USUAL ORDER)

$$\begin{array}{c} 1\\2\\3\\4\\5\\6\\7 \end{array}
\begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\\ v_7 \end{bmatrix}$$

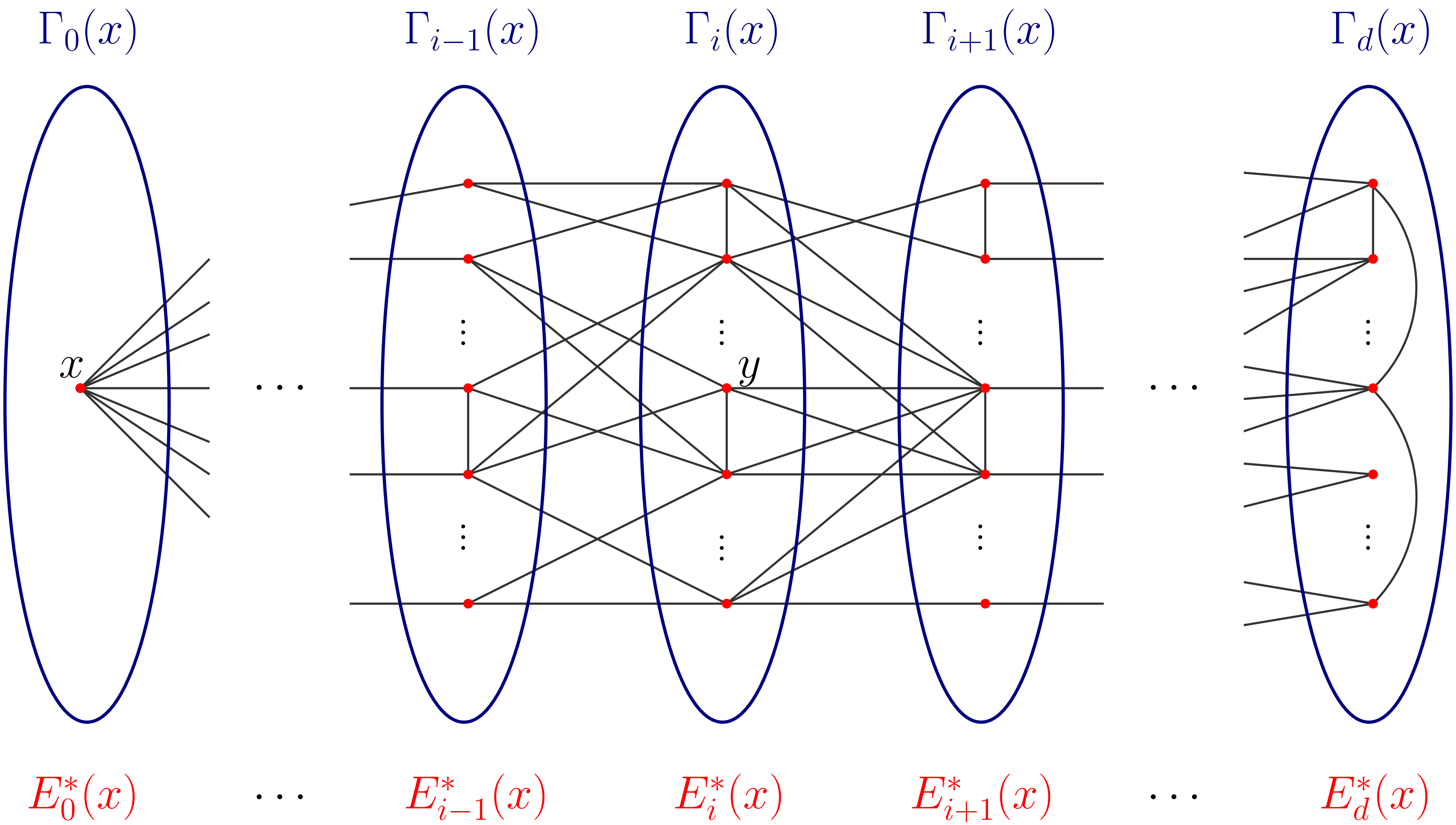
STANDARD MODULE

$$V = \mathbb{C}^7$$

$$\begin{array}{c} 1\\2\\3\\4\\5\\6\\7 \end{array}
\begin{bmatrix} 1\\0\\0\\0\\0\\0\\0 \end{bmatrix}
\qquad
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7 \end{array}
\begin{bmatrix} 0\\0\\0\\1\\0\\0\\0 \end{bmatrix}
\qquad
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7 \end{array}
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\1 \end{bmatrix}$$

$$\hat{1} \qquad \hat{4} \qquad \hat{7}$$

The dual idempotents of Γ



Terminology and Notations

Let $\Gamma = (X, \mathcal{R})$ be a graph and let $\mathbb{C}$ denote the complex number field.

- Fix $x \in X$. For every integer i, $0 \le i \le \epsilon(x)$, the ***i*-th dual idempotent of Γ with respect to** x is the diagonal matrix $E_i^* := E_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry given as follows:

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if} \quad \partial(x, y) = i, \\ 0 & \text{if} \quad \partial(x, y) \neq i. \end{cases}$$

- The **adjacency matrix of** Γ is the matrix $A := A(\Gamma) \in \mathrm{Mat}_X(\mathbb{C})$ where, for every $x, y \in X$, the (x, y)-entry is defined as follows:

$$A_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x, y) = 1, \\ 0 & \text{if} \quad \partial(x, y) \neq 1. \end{cases}$$

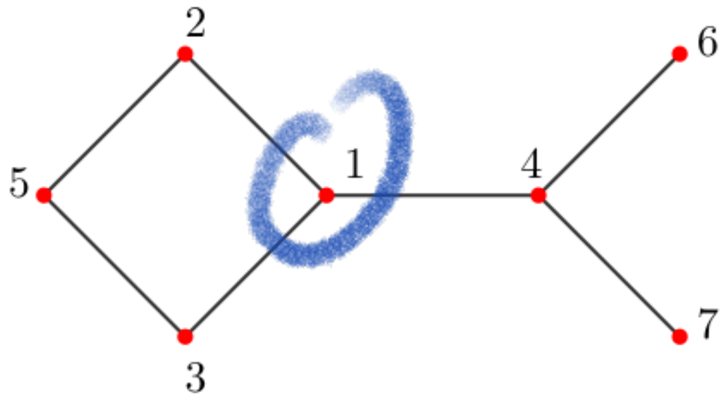
Terminology and Notations

Definition (Terwilliger, 1992.)

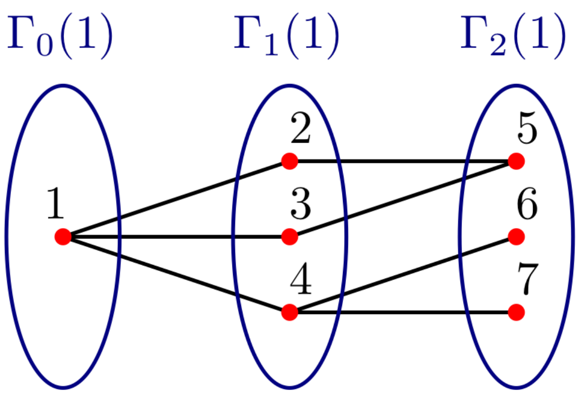
The **Terwilliger algebra of Γ with respect to** x is considered to be the subalgebra $T := T(x)$ of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ and the dual idempotents of Γ with respect to x:

$$T := \left\langle A, E_0^*, \cdots, E_i^*, \cdots, E_{\epsilon(x)}^* \right\rangle .$$

Algebra T is finite dimensional and semisimple.



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$



TERWILLIGER ALGEBRA OF Γ WITH RESPECT TO 1

$$T(1) = \langle A, E_0^*(1), E_1^*(1), E_2^*(1) \rangle$$

$$E_0^*(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad E_1^*(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad E_2^*(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Terminology and Notations

Let Γ be a graph and let $T := T(x)$ the Terwilliger algebra of Γ with respect to x.

- A **T-module** is a subspace W of V which is B-invariant for every $B \in T$.
- A **T-module** W is **irreducible** whenever W is non-zero and W contains no T-modules other than 0 and W.
- It turns out that any T-module is orthogonal direct sum of irreducible T-modules.

Terminology and Notations

Let Γ be a graph and let $T := T(x)$ the Terwilliger algebra of Γ with respect to x.

- Each irreducible T-module W is orthogonal direct sum of the non-vanishing subspaces $E_i^* W$, where $0 \leq i \leq \epsilon(x)$:

$$W = E_0^* W + \cdots + E_i^* W + \cdots + E_{\epsilon(x)}^* W.$$

- A scalar $r := r(W)$ is the **endpoint of** W if

$$r = \min \{ i : 0 \leq i \leq \epsilon(x), E_i^* W \neq 0 \} .$$

- The T-module W is called **thin** whenever the dimension of $E_i^* W$ is at most 1 for $0 \leq i \leq \epsilon(x)$.

Irreducible T-modules with endpoint 0

The trivial T-module

Let T be the Terwilliger algebra of Γ with respect to x. Let W denote an irreducible T-module with endpoint 0. Then,

- There exists a nonzero vector $w \in E_0^* W$.

  THEN $w = E_0^* v$ FOR SOME $v \in W$. THEREFORE,

  $$E_0^* w = E_0^* E_0^* v = E_0^* v = w \quad (E_0^* E_0^* = E_0^*)$$

- We know that $w = \sum_{y \in X} \alpha_y \, \widehat{y}$ for some scalars $\alpha_y \in \mathbb{C}$.

  $$w = E_0^* w = \sum_{y \in X} \alpha_y . E_0^* \hat{y} = \alpha_x . E_0^* \hat{x} = \alpha_x . \hat{x}$$

- We thus have $w = E_0^* w = \alpha_x \, \widehat{x}$ with $\alpha_x \neq 0$.

- This shows $\widehat{x} = \alpha_x^{-1} E_0^* w$ and so, $\widehat{x} \in E_0^* W \subseteq W$.

$$\hat{x} = E_0^* \left(\frac{1}{\alpha_x} \cdot w \right) \in E_0^* W$$

$\frac{1}{\alpha_x} \cdot w \in W$

- Since W is a T-module, W is B-invariant for every $B \in T$.

$\hat{x} \in W \Rightarrow B\hat{x} \in W$ FOR ALL $B \in T$

- This implies that $B\widehat{x} \in W$ for every matrix $B \in T$.
- The set $T\hat{x} := \{B\hat{x} : B \in T\} \subseteq W$ is a non-zero T-module.

THEREFORE,

$$T\hat{x} = W.$$

DEFINITION

The **trivial T-module** is the unique irreducible T-module with endpoint 0. Namely, the set $T\hat{x} := \{B\hat{x} : B \in T\}$.

The lowering, flat and raising matrices

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x. Let $d = \epsilon(x)$.

Definition

Define matrices $L = L(x)$, $F = F(x)$ and $R = R(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by

$$L = \sum_{i=1}^{d} E^*_{i-1} A E^*_i, \qquad F = \sum_{i=0}^{d} E^*_i A E^*_i, \qquad R = \sum_{i=0}^{d-1} E^*_{i+1} A E^*_i.$$

We refer to L, F and R as the **lowering**, the **flat** and the **raising matrix with respect to** x, respectively. Note that $L, F, R \in T$. Moreover, we have that $F = F^\top$, $R = L^\top$ and $A = L + F + R$.

The lowering, flat and raising matrices

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Rows and columns: $\Gamma_0(1)$, $\Gamma_1(1)$, $\Gamma_2(1)$

$$L = E_0^* . A . E_1^* + E_1^* . A . E_2^*$$

$$L = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = L + F + R$$

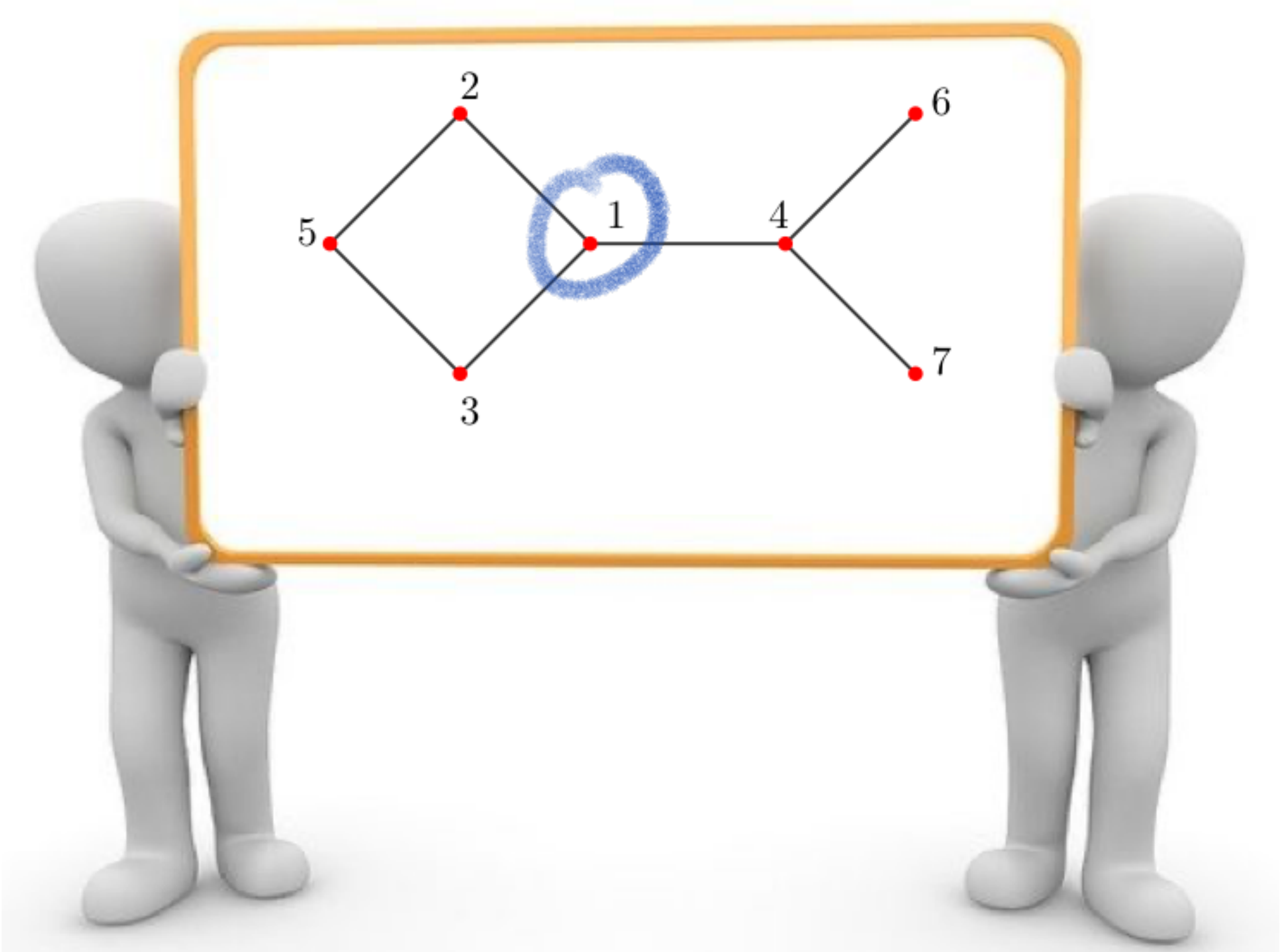
$$F = E_0^* . A . E_0^* + E_1^* . A . E_1^* + E_2^* . A . E_2^*$$

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R = E_1^* . A . E_0^* + E_2^* . A . E_1^*$$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Example: study the module $\widehat{T1}$



Vectors $R^i \hat{1}$

$$R^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$R^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R^m = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$m \geq 3$

Vectors $R^i\hat{1}$ $(0 \le i \le 2)$

LET W DENOTE THE SUBSPACE OF $\mathbb{C}^7$ GENERATED BY THE VECTORS $R^i\hat{1}$ $(0 \le i \le 2)$

$$\hat{1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad R\hat{1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad R^2\hat{1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

$$W = \operatorname{span}\{\hat{1}, R\hat{1}, R^2\hat{1}\} \subseteq T\hat{1}$$

RECALL THAT $T\hat{1} = \{B\hat{1} : B \in T\}$

$E_i^* W \subseteq W$ $(0 \le i \le 2)$ and $AW \subseteq W$

$$W = \text{span}\{\hat{1}, R\hat{1}, R^2\hat{1}\} \subseteq T\hat{1}$$

$$E_0^*\hat{1} = \hat{1} \qquad E_0^* R\hat{1} = 0 \qquad E_0^* R^2\hat{1} = 0$$

$$E_1^*\hat{1} = 0 \qquad E_1^* R\hat{1} = R\hat{1} \qquad E_1^* R^2\hat{1} = 0$$

$$E_2^*\hat{1} = 0 \qquad E_2^* R\hat{1} = 0 \qquad E_2^* R^2\hat{1} = R^2\hat{1}$$

$$E_i^* W \subseteq W \quad (0 \le i \le 2)$$

$$A\hat{1} = R\hat{1}$$

$$AR\hat{1} = 3.\hat{1} + R^2\hat{1}$$

$$AR^2\hat{1} = 2.R\hat{1}$$

$$AW \subseteq W$$

Example: study the module $T\hat{1}$

WE THUS HAVE $E_i^* W \subseteq W \quad (0 \le i \le 2)$

$AW \subseteq W$

THIS SHOWS THAT W IS A T-MODULE

SINCE $W \neq 0$ AND $W \subseteq T\hat{1}$

WE MUST HAVE $W = T\hat{1}$.

THEREFORE, W IS THE UNIQUE IRREDUCIBLE T-MODULE WITH ENDPOINT 0.

Example: study the module $T\hat{1}$

$$W = \text{span}\{\hat{1}, R\hat{1}, R^2\hat{1}\} = T\hat{1}$$

$$E_i^* W = \text{span}\{E_i^*\hat{1}, E_i^* R\hat{1}, E_i^* R^2\hat{1}\}$$

$E_0^*\hat{1} = \hat{1}$ | $E_0^* R\hat{1} = 0$ | $E_0^* R^2\hat{1} = 0$

$E_1^*\hat{1} = 0$ | $E_1^* R\hat{1} = R\hat{1}$ | $E_1^* R^2\hat{1} = 0$

$E_2^*\hat{1} = 0$ | $E_2^* R\hat{1} = 0$ | $E_2^* R^2\hat{1} = R^2\hat{1}$

$$E_0^* W = \text{span}\{\hat{1}\}$$

$$E_1^* W = \text{span}\{R\hat{1}\}$$

$$E_2^* W = \text{span}\{R^2\hat{1}\}$$

$\dim(E_i^* W) = 1 \quad (0 \le i \le 2)$

W IS THIN

Let $\Gamma = (X, \mathscr{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

Question

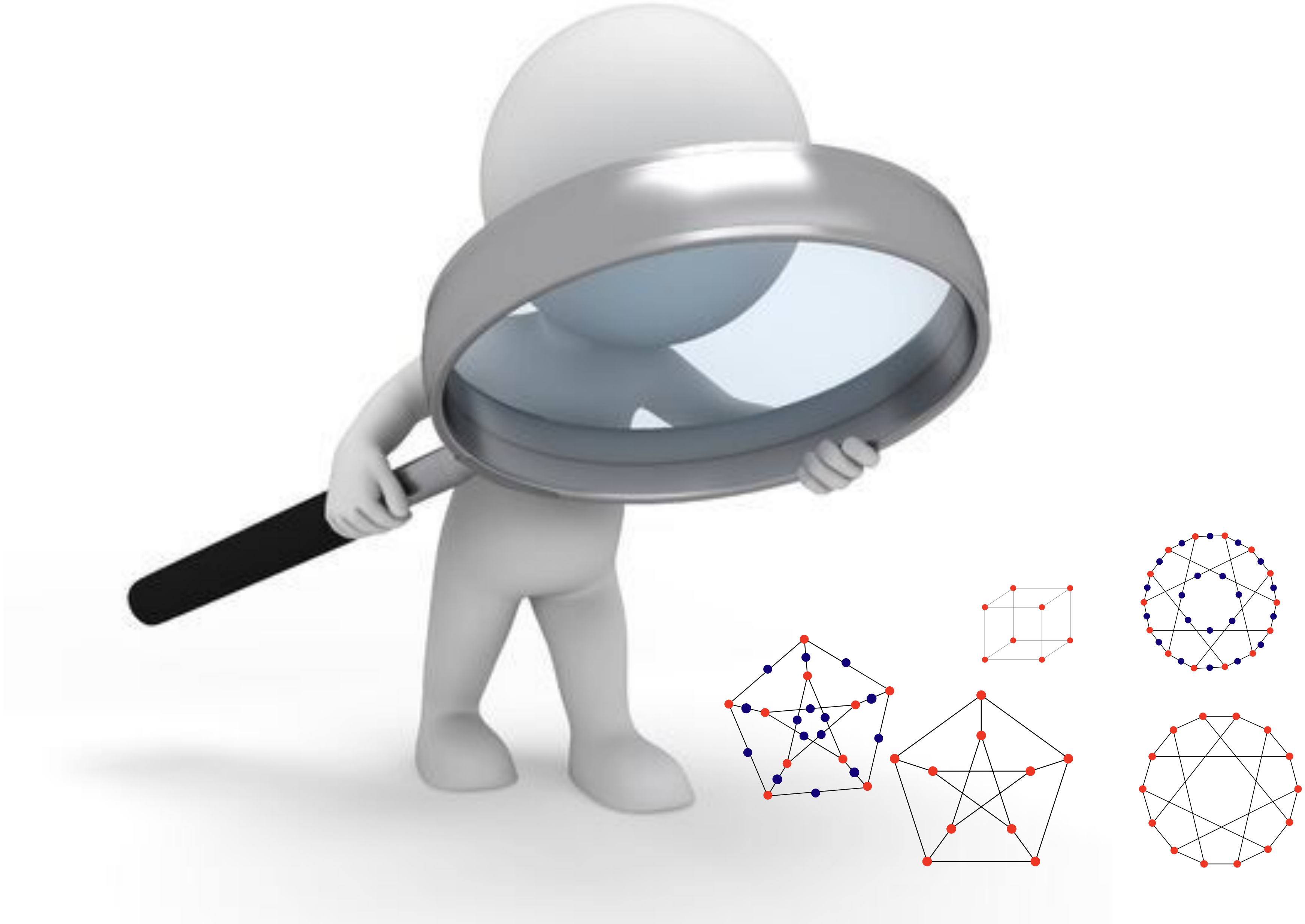
How the module structure of T and certain combinatorial properties of Γ are related?

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

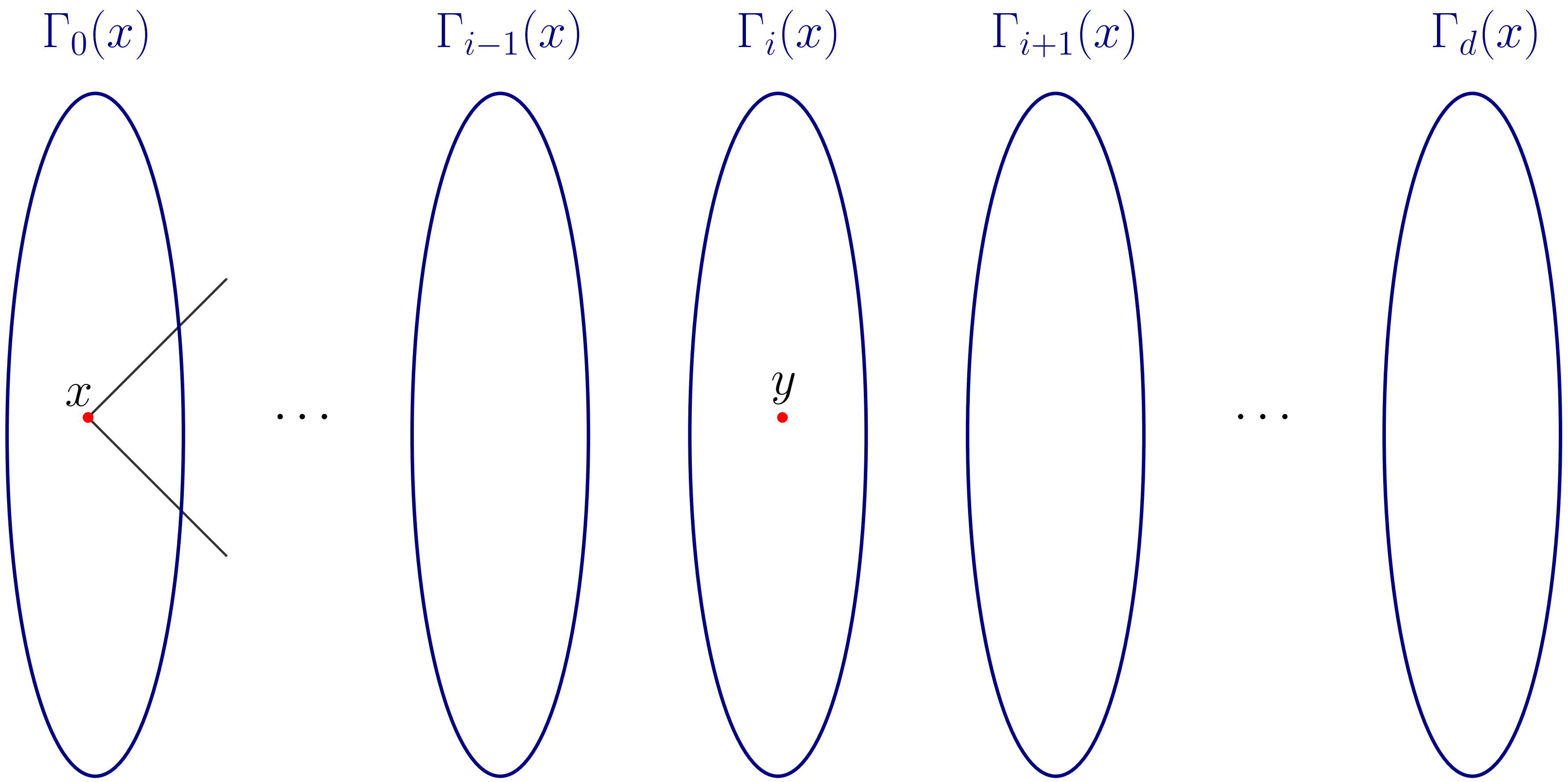
QUESTION

Is there any way to study the trivial module $T\widehat{x}$ from a combinatorial point of view?

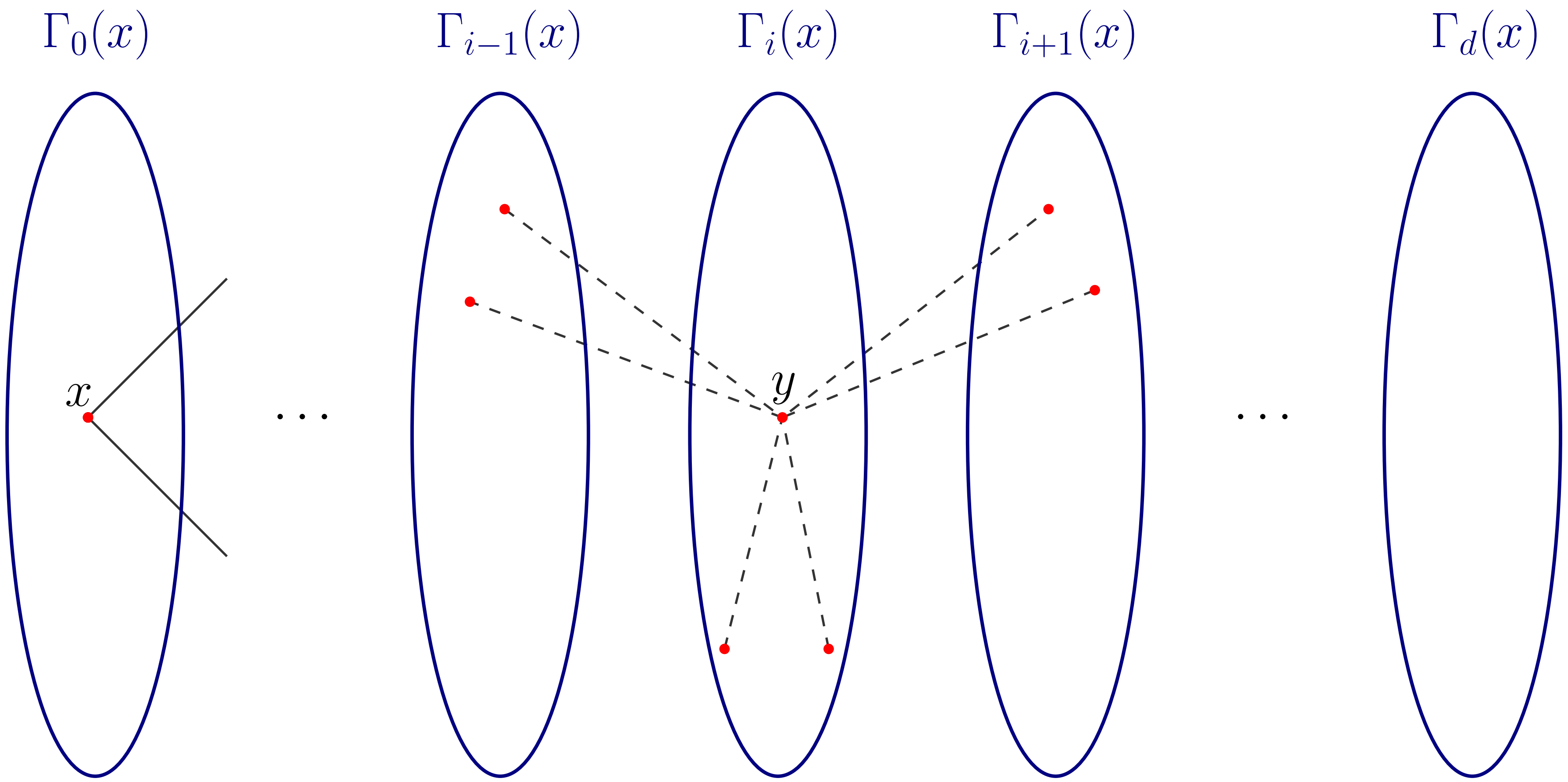
Local Distance-Regularity



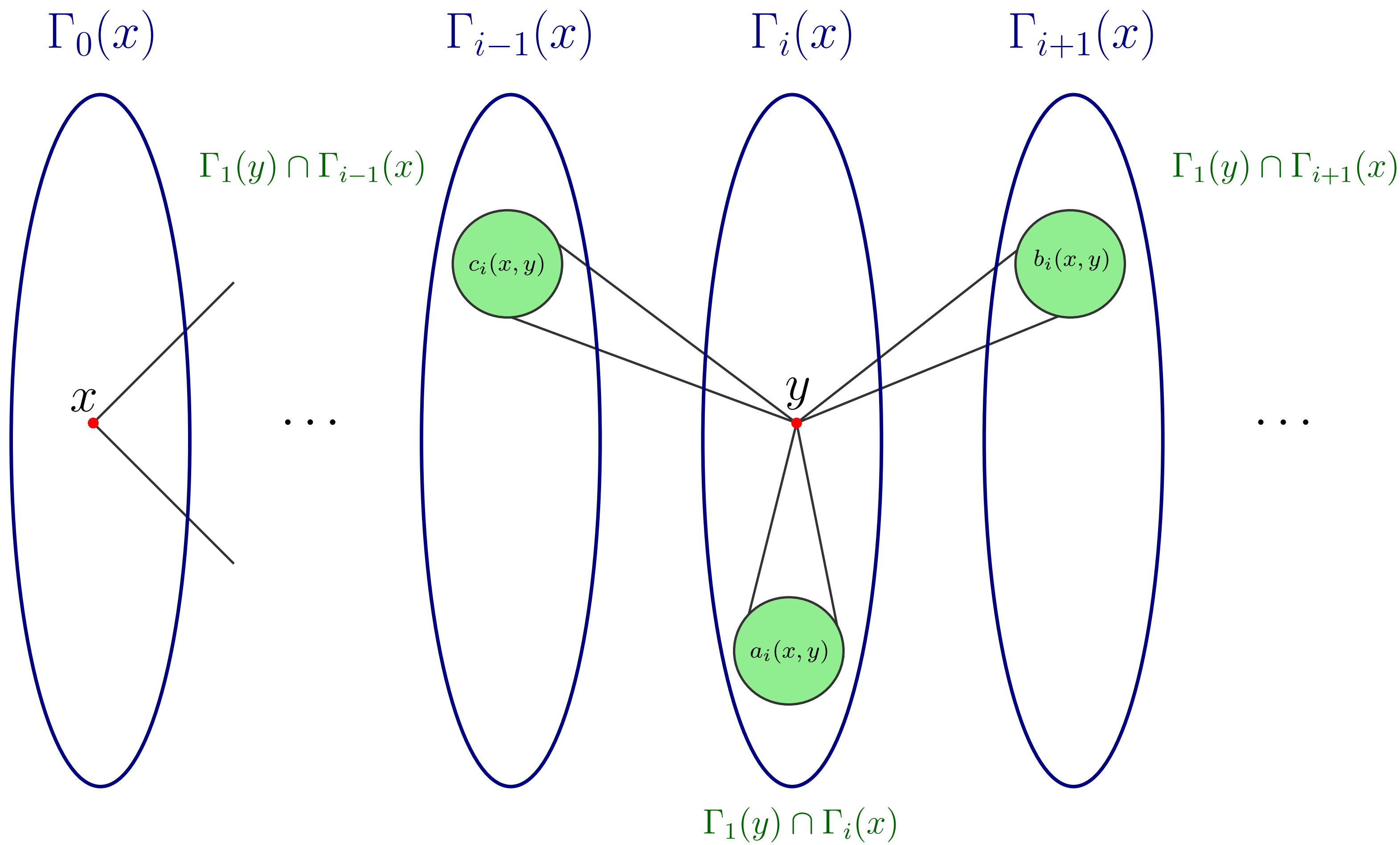
Local Distance-Regularity



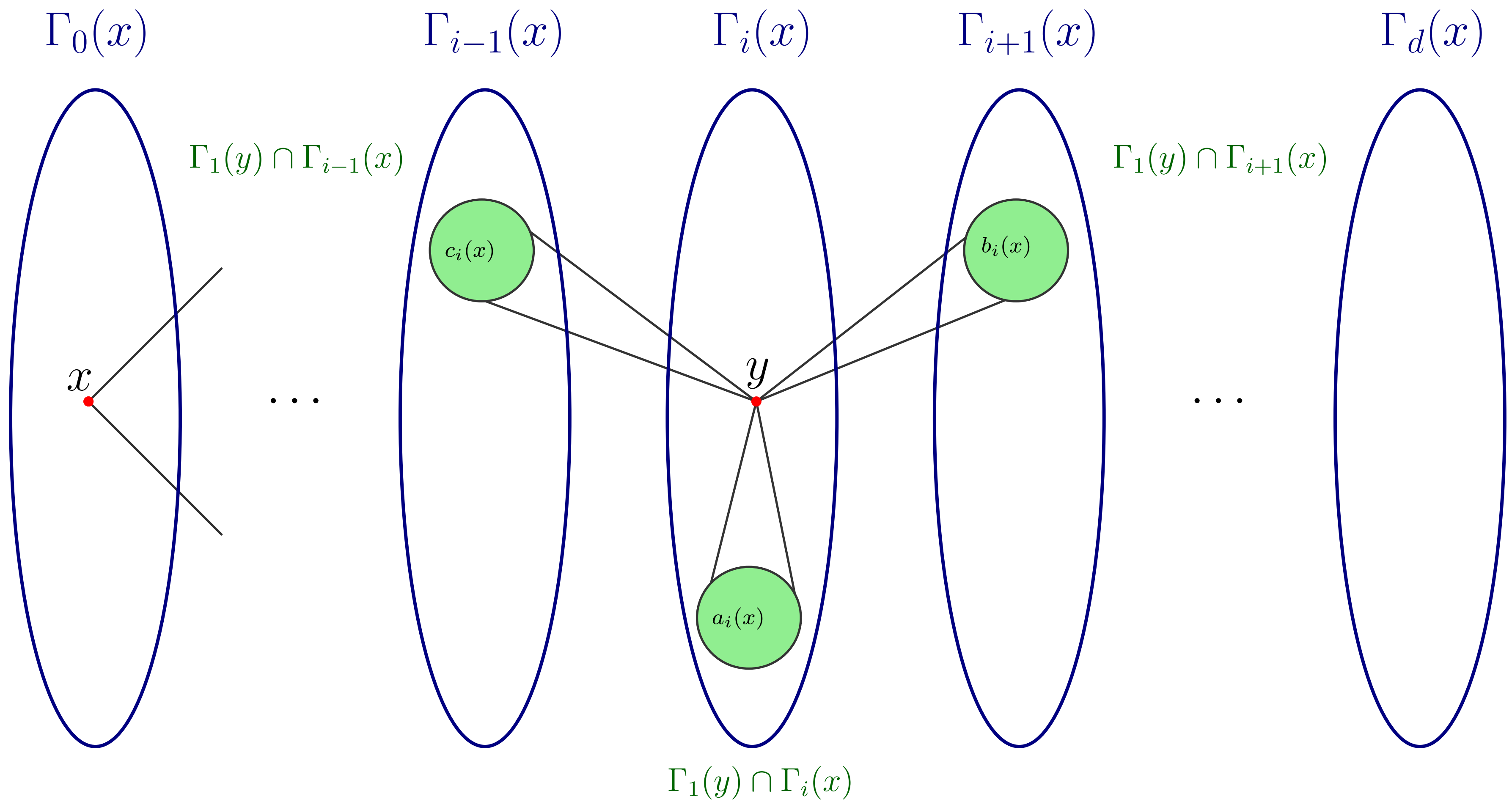
Local Distance-Regularity



Local Distance-Regularity



Local Distance-Regularity



Local Distance-Regularity

Pick $x \in X$. Suppose that $y \in \Gamma_i(x)$ for some $0 \leq i \leq \epsilon(x)$. Then, the following numbers are defined:

$$a_i(x, y) := |\Gamma_i(x) \cap \Gamma_1(y)|, \qquad b_i(x, y) := |\Gamma_{i+1}(x) \cap \Gamma_1(y)|,$$

$$c_i(x, y) := |\Gamma_{i-1}(x) \cap \Gamma_1(y)|.$$

Definition (Godsil and Shawe-Taylor, 1987.)

A vertex $x \in X$ is **distance-regularized** (or Γ is **distance-regular around** x), if for $x \in X$, the numbers $a_i(x, y)$, $b_i(x, y)$ and $c_i(x, y)$ do not depend on the choice of $y \in \Gamma_i(x)$ for every $0 \leq i \leq \epsilon(x)$.

In this case, the numbers $a_i(x, y)$, $b_i(x, y)$ and $c_i(x, y)$ are simply denoted by $a_i(x)$, $b_i(x)$ and $c_i(x)$ respectively, and are called the **intersection numbers of** x.

Local Distance-Regularity

Definition

- A **distance-regularized graph** is considered to be a connected graph in which every vertex is distance-regularized.
- A **distance-regular graph** is a distance-regularized graph where all its vertices have the same intersection numbers.
- A distance-regularized graph is said to be **distance-biregular** if the following hold:
  - It is bipartite.
  - Vertices in the same color partition have the same intersection numbers.
  - Vertices in different color partitions have different intersection numbers.

Theorem (Godsil and Shawe-Taylor, 1985)

Every distance-regularized graph is either distance-regular or distance-biregular.

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

Question

Is there any relation between local distance-regularity and the trivial T-modules of Γ ?

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

THEOREM (TERWILLIGER, 1993)

If Γ is distance-regular around x then the T-module $T\hat{x}$ is thin.

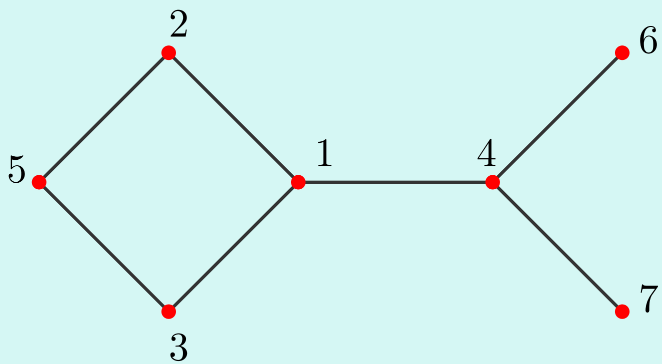
Let $\Gamma = (X, \mathscr{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

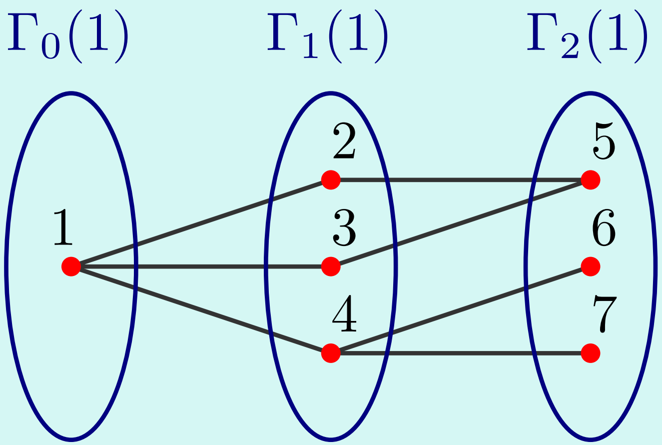
Question

If the trivial T-module $T\hat{x}$ is thin, is Γ distance-regular around x?

Example

Let $\Gamma = (X, \mathscr{R})$ be the simple graph with $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathscr{R} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{3, 5\}, \{4, 6\}, \{4, 7\}\}$.





The trivial $T\widehat{1}$ is thin. However, Γ is not distance-regular around 1.

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

Theorem (Terwilliger, 1993)

The following statements are equivalent:

1. Γ is distance-regularized.
2. For every $x \in X$, the collection of all the vectors

$$s_i(x) = \sum_{y \in \Gamma_i(x)} \widehat{y} \qquad (0 \leq i \leq \epsilon(x)),$$

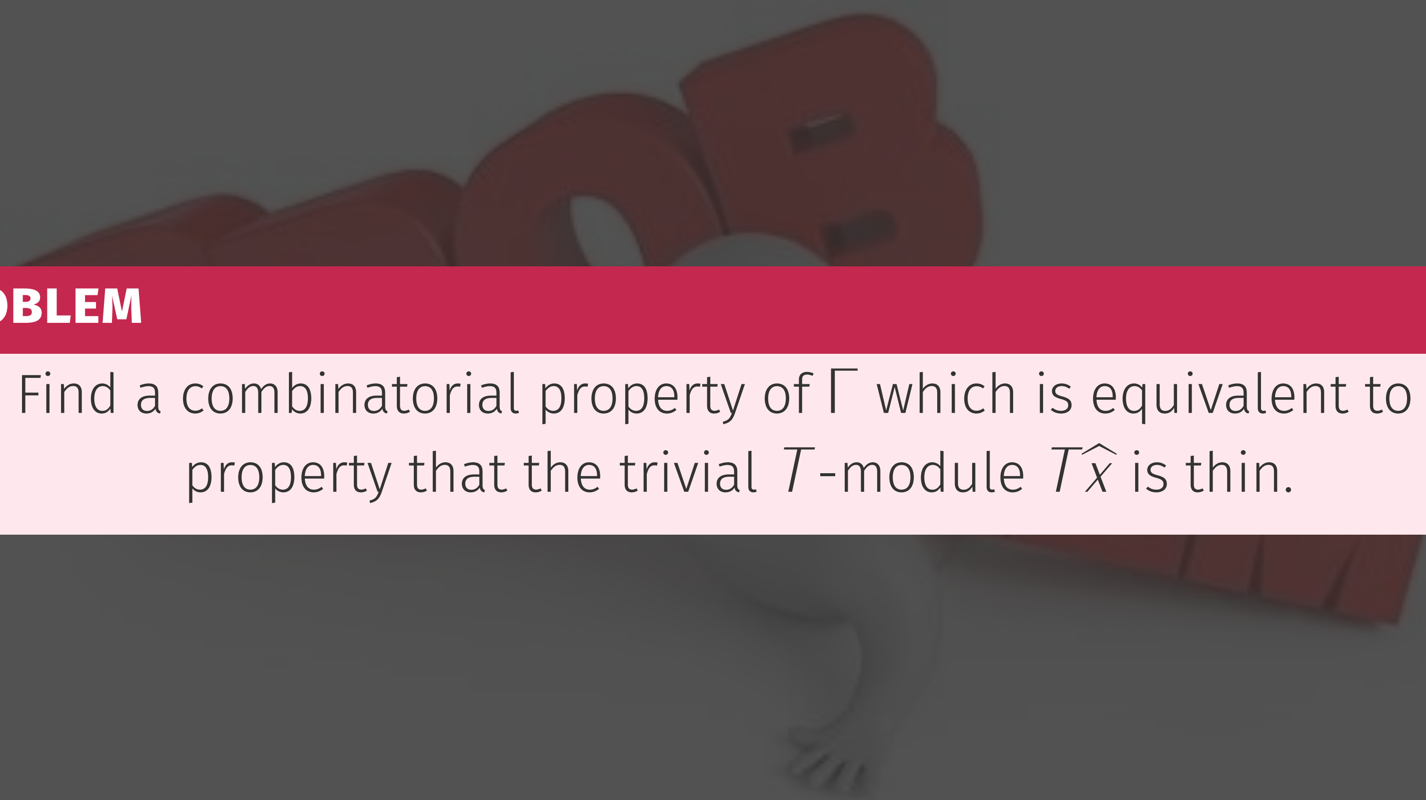
   is a basis of the trivial $T(x)$-module.

3. For every $x \in X$, the trivial $T(x)$-module is thin.

If the above conditions hold, then Γ is either distance-regular or distance-biregular.

Main Motivation

Main Motivation



Problem

Find a combinatorial property of Γ which is equivalent to the property that the trivial T-module $T\widehat{x}$ is thin.

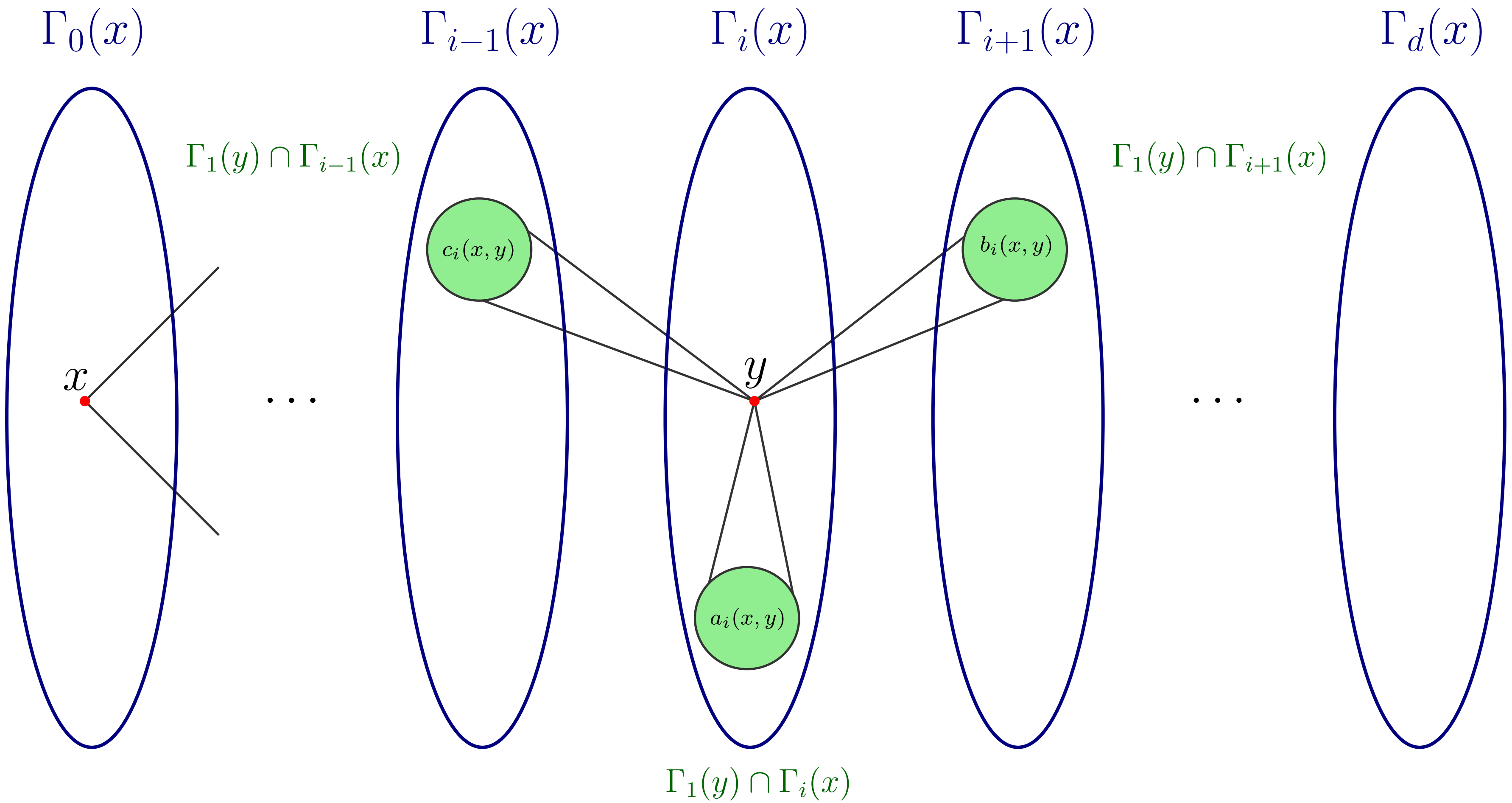
Local Pseudo Distance-Regularity

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

Theorem (Fiol, Garriga, 1999)

The trivial T-module $T\hat{x}$ is thin if and only if Γ is pseudo distance-regular around x.

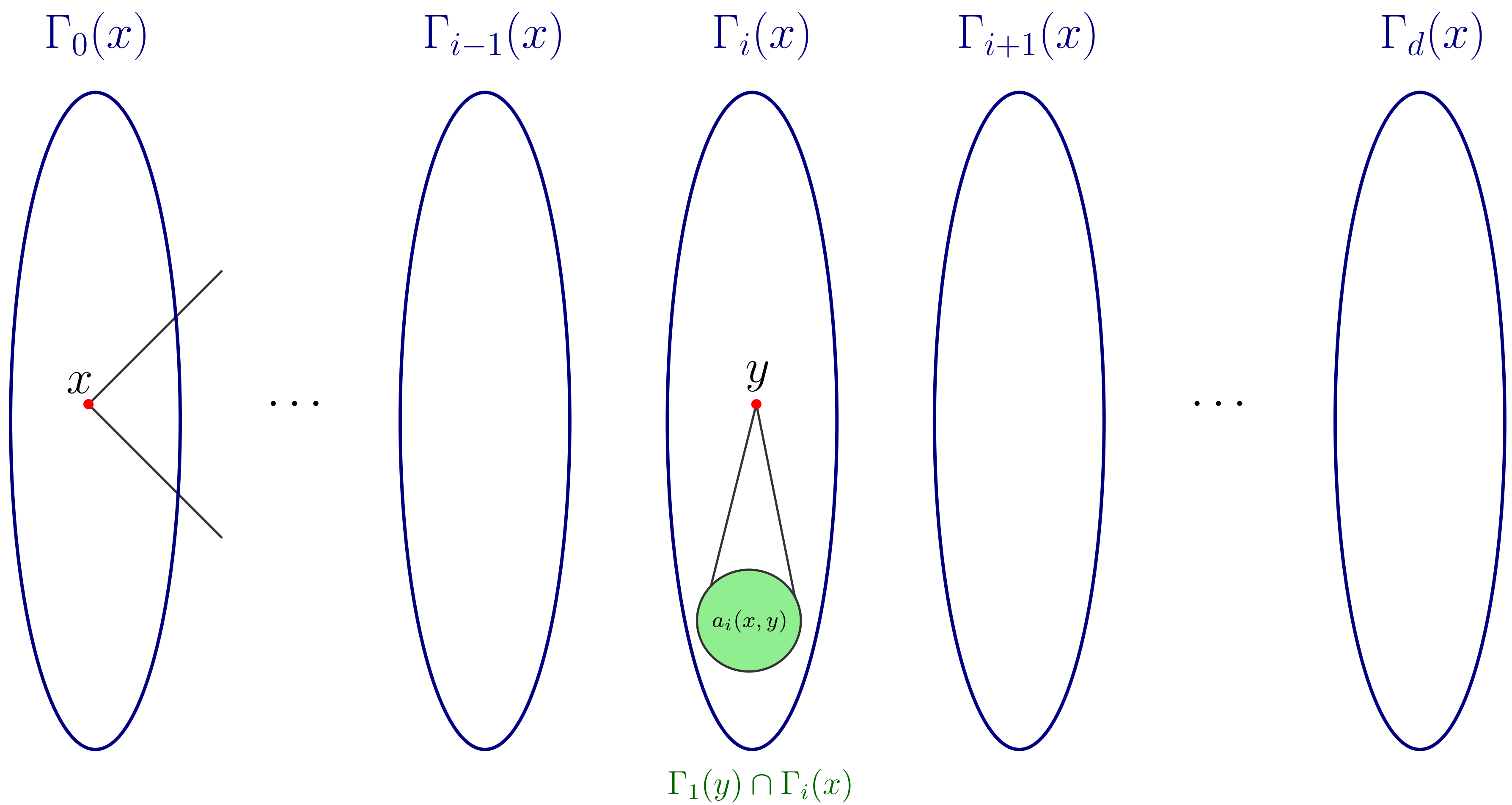
Local Pseudo Distance-Regularity



Let $\upsilon = (\upsilon_x, \ldots, \upsilon_y, \ldots, \upsilon_z)^t$ be a Perron-Frobenius vector of A.

Local Pseudo Distance-Regularity

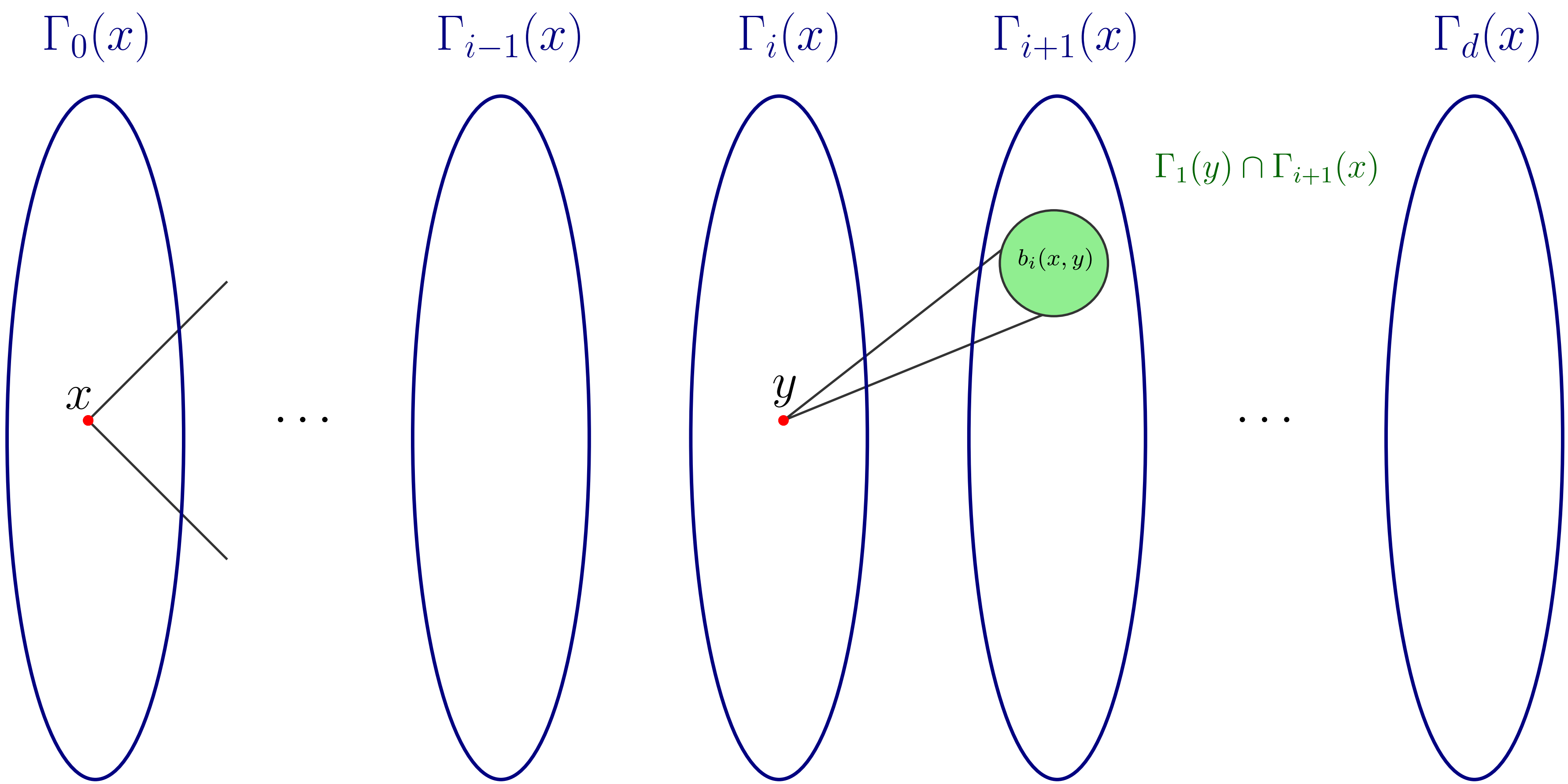
$$\upsilon = (\upsilon_x, \ldots, \upsilon_y, \ldots, \upsilon_z)^t$$



$$a_i^*(x,y) := \sum_{z \in \Gamma_1(y) \cap \Gamma_i(x)} \frac{\upsilon_z}{\upsilon_y}$$

Local Pseudo Distance-Regularity

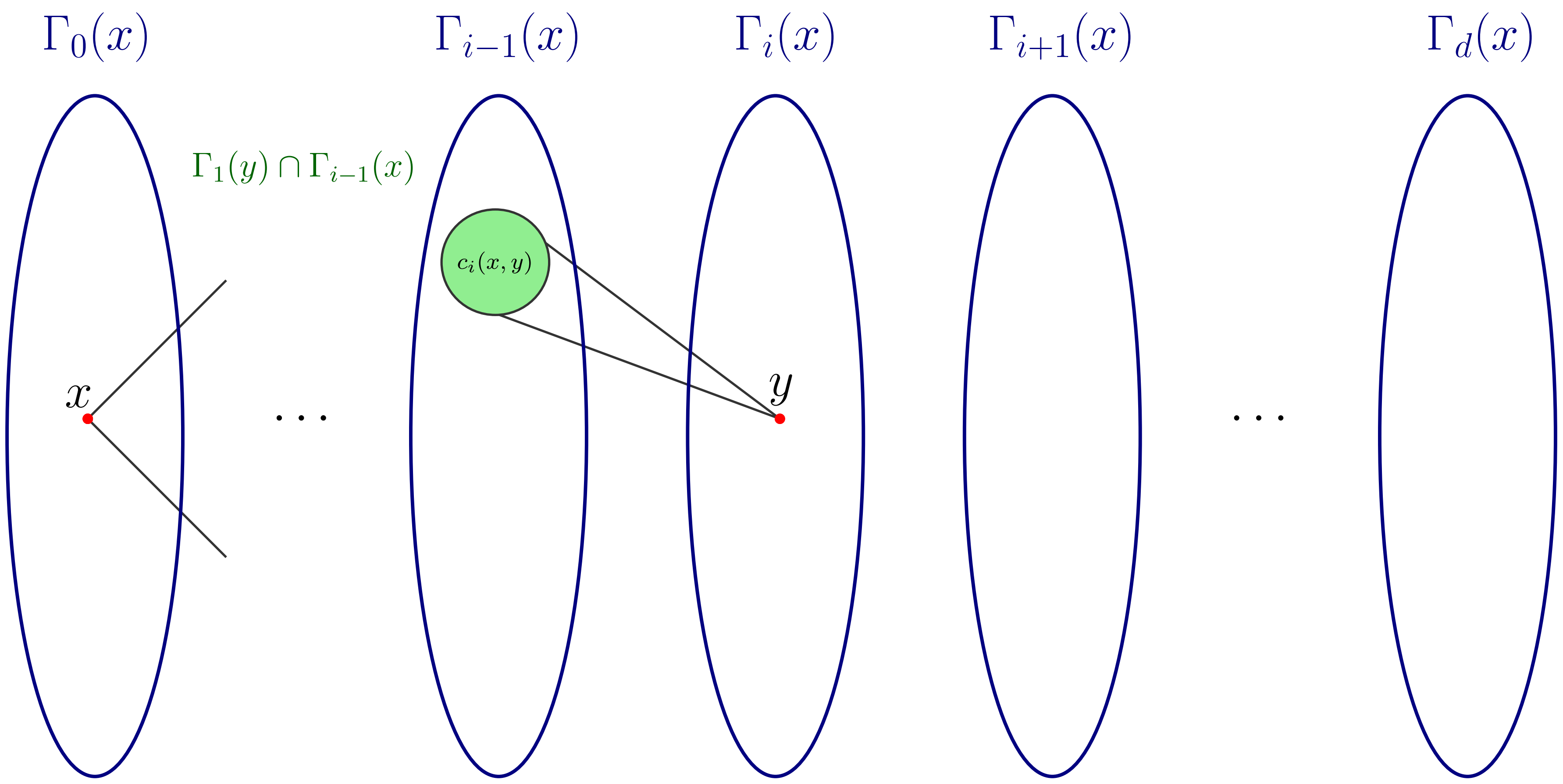
$$\upsilon = (\upsilon_x, \ldots, \upsilon_y, \ldots, \upsilon_z)^t$$



$$b_i^*(x,y) := \sum_{z \in \Gamma_1(y) \cap \Gamma_{i+1}(x)} \frac{\upsilon_z}{\upsilon_y}$$

Local Pseudo Distance-Regularity

$$\upsilon = (\upsilon_x, \ldots, \upsilon_y, \ldots, \upsilon_z)^t$$



$$c_i^*(x,y) := \sum_{z \in \Gamma_1(y) \cap \Gamma_{i-1}(x)} \frac{\upsilon_z}{\upsilon_y}$$

Local Pseudo Distance-Regularity

Pick $x \in X$. Suppose that $y \in \Gamma_i(x)$ for some $0 \leq i \leq \epsilon(x)$. Then, the following numbers are defined:

$$a_i^*(x,y) := \sum_{z \in \Gamma_1(y) \cap \Gamma_i(x)} \frac{\upsilon_z}{\upsilon_y}, \qquad b_i^*(x,y) := \sum_{z \in \Gamma_1(y) \cap \Gamma_{i+1}(x)} \frac{\upsilon_z}{\upsilon_y},$$

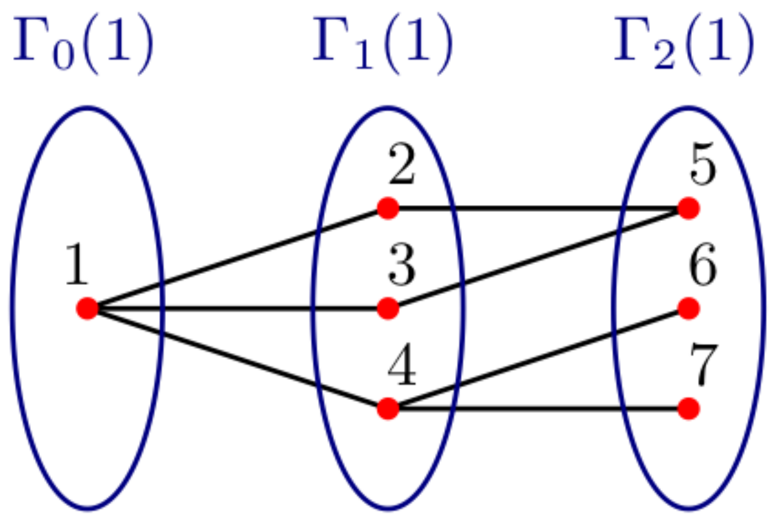
$$c_i^*(x,y) := \sum_{z \in \Gamma_1(y) \cap \Gamma_{i-1}(x)} \frac{\upsilon_z}{\upsilon_y}.$$

Definition (Fiol, Garriga and Yebra, 1996.)

A vertex $x \in X$ is **pseudo-distance-regularized** (or Γ is **pseudo-distance-regular around** x), if for $x \in X$, the numbers $a_i^*(x,y)$, $b_i^*(x,y)$ and $c_i^*(x,y)$ do not depend on the choice of $y \in \Gamma_i(x)$ for every $0 \leq i \leq \epsilon(x)$.

Example

Consider $v = (3\ \sqrt{5}\ \sqrt{5}\ \sqrt{5}\ 2\ 1\ 1)^t$. Let's find the value $b_1^*(1)$.



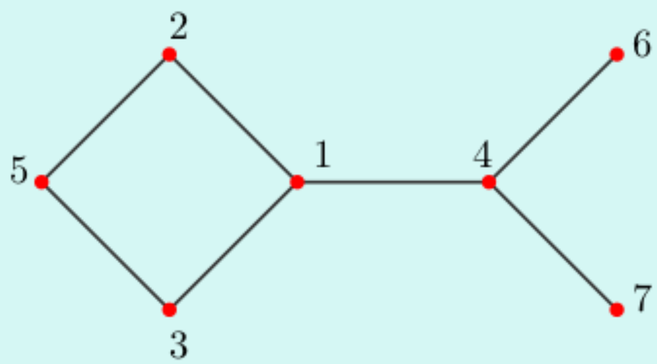
$$b_1^*(1,2) := \frac{v_5}{v_2} = \frac{2}{\sqrt{5}}, \qquad b_1^*(1,3) := \frac{v_5}{v_3} = \frac{2}{\sqrt{5}},$$

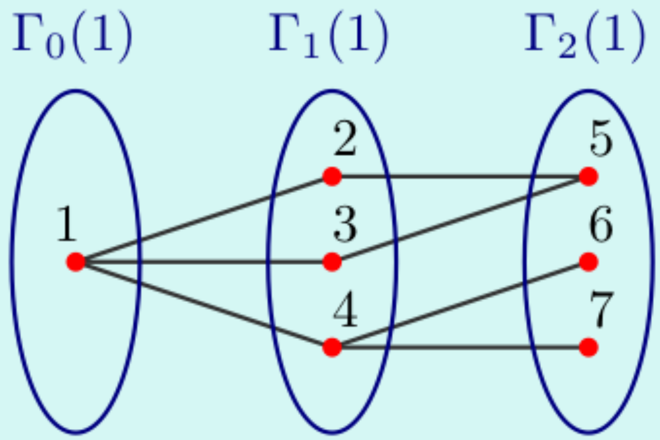
$$b_1^*(1,4) := \frac{v_6 + v_7}{v_4} = \frac{2}{\sqrt{5}}.$$

$$\boxed{b_1^*(1) = \frac{2}{\sqrt{5}}}$$

Example

Let $\Gamma = (X, \mathscr{R})$ be the simple graph with $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathscr{R} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{3, 5\}, \{4, 6\}, \{4, 7\}\}$.





Γ is pseudo distance-regular around 1.

The trivial module $T\hat{1}$ is thin.

Γ is not distance-regular around 1.

Theorem (Fiol, Garriga, 1999)

The trivial T-module $T\hat{x}$ is thin if and only if Γ is pseudo distance-regular around x.

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

PROBLEM

Find a **PURELY** combinatorial property of Γ which is equivalent to the property that the trivial T-module $T\hat{x}$ is thin.

The shape of a walk with respect to a given vertex

Let $\Gamma = (X, \mathscr{R})$ denote a finite, simple and connected graph.

Definition

Pick $x, y, z \in X$ and let $P = [y = x_0, x_1, \ldots, x_j = z]$ denote a yz-walk. The **shape of P with respect to** x is a sequence of symbols $t_1 t_2 \ldots t_j$, where $t_i \in \{f, \ell, r\}$, and such that

$$t_i = \begin{cases} r & \text{if} \quad \partial(x, x_i) = \partial(x, x_{i-1}) + 1, \\ f & \text{if} \quad \partial(x, x_i) = \partial(x, x_{i-1}), \\ \ell & \text{if} \quad \partial(x, x_i) = \partial(x, x_{i-1}) - 1, \end{cases} \qquad (1 \leq i \leq j).$$

We use exponential notation for shapes containing several consecutive identical symbols. For instance, instead of $rrrrfff\ell\ell r$ we simply write $r^4 f^3 \ell^2 r$.

The lowering, flat and raising matrices

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x. Let $d = \epsilon(x)$.

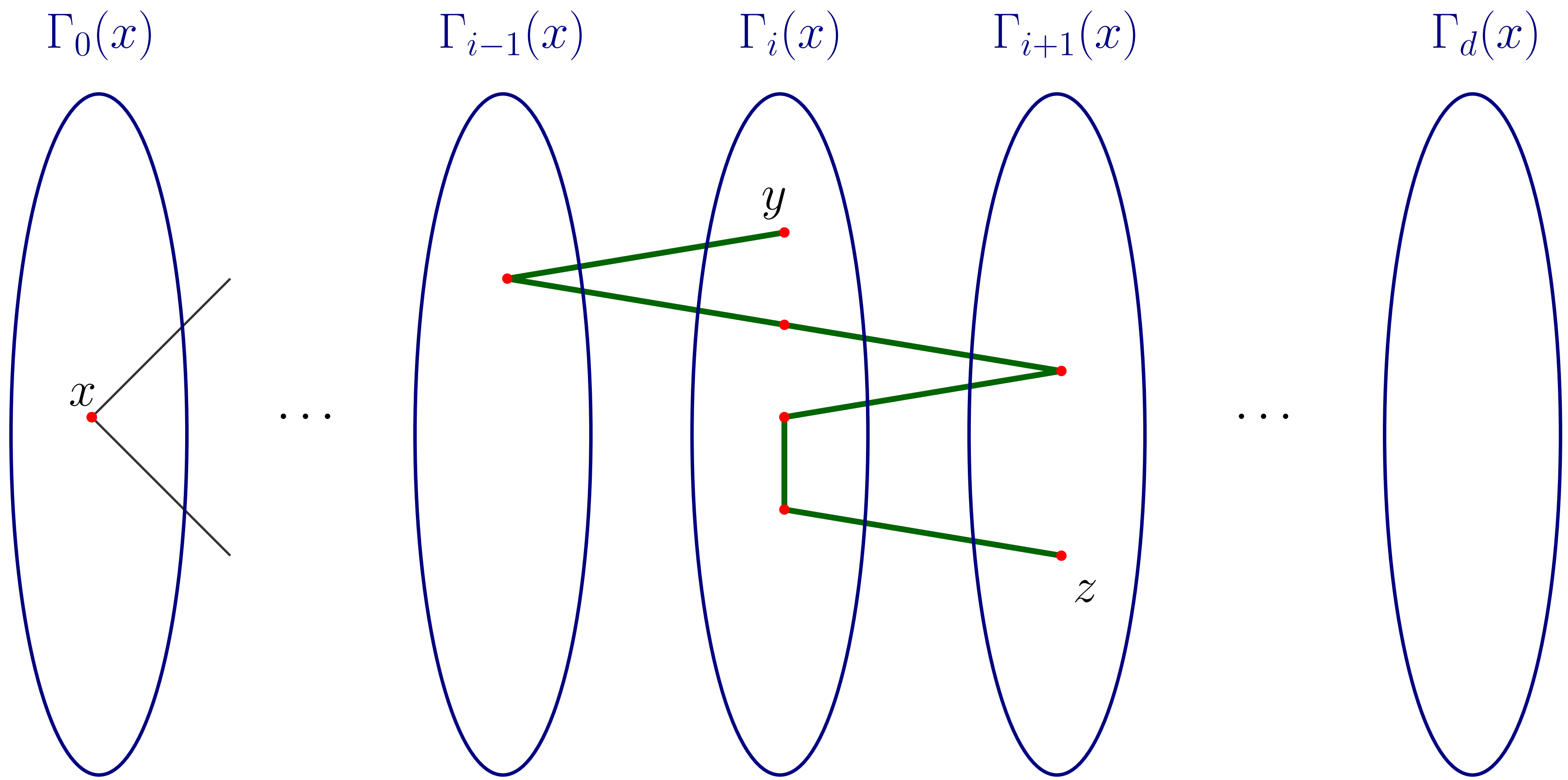
DEFINITION

Define matrices $L = L(x)$, $F = F(x)$ and $R = R(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by

$$L = \sum_{i=1}^{d} E^*_{i-1} A E^*_i, \qquad F = \sum_{i=0}^{d} E^*_i A E^*_i, \qquad R = \sum_{i=0}^{d-1} E^*_{i+1} A E^*_i.$$

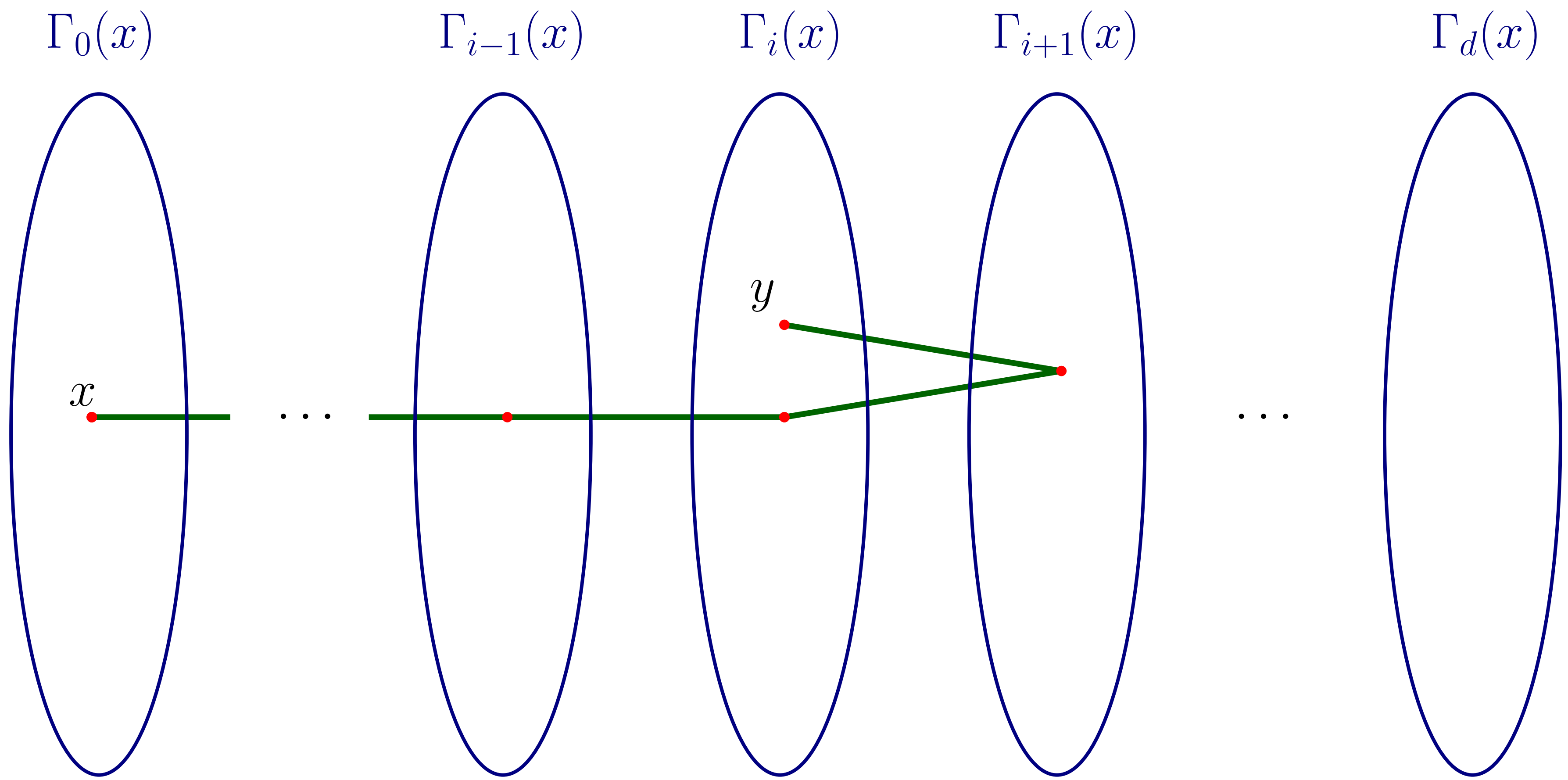
We refer to L, F and R as the **lowering**, the **flat** and the **raising matrix with respect to** x, respectively. Note that $L, F, R \in T$. Moreover, we have that $F = F^\top$, $R = L^\top$ and $A = L + F + R$.

Walks of a certain shape



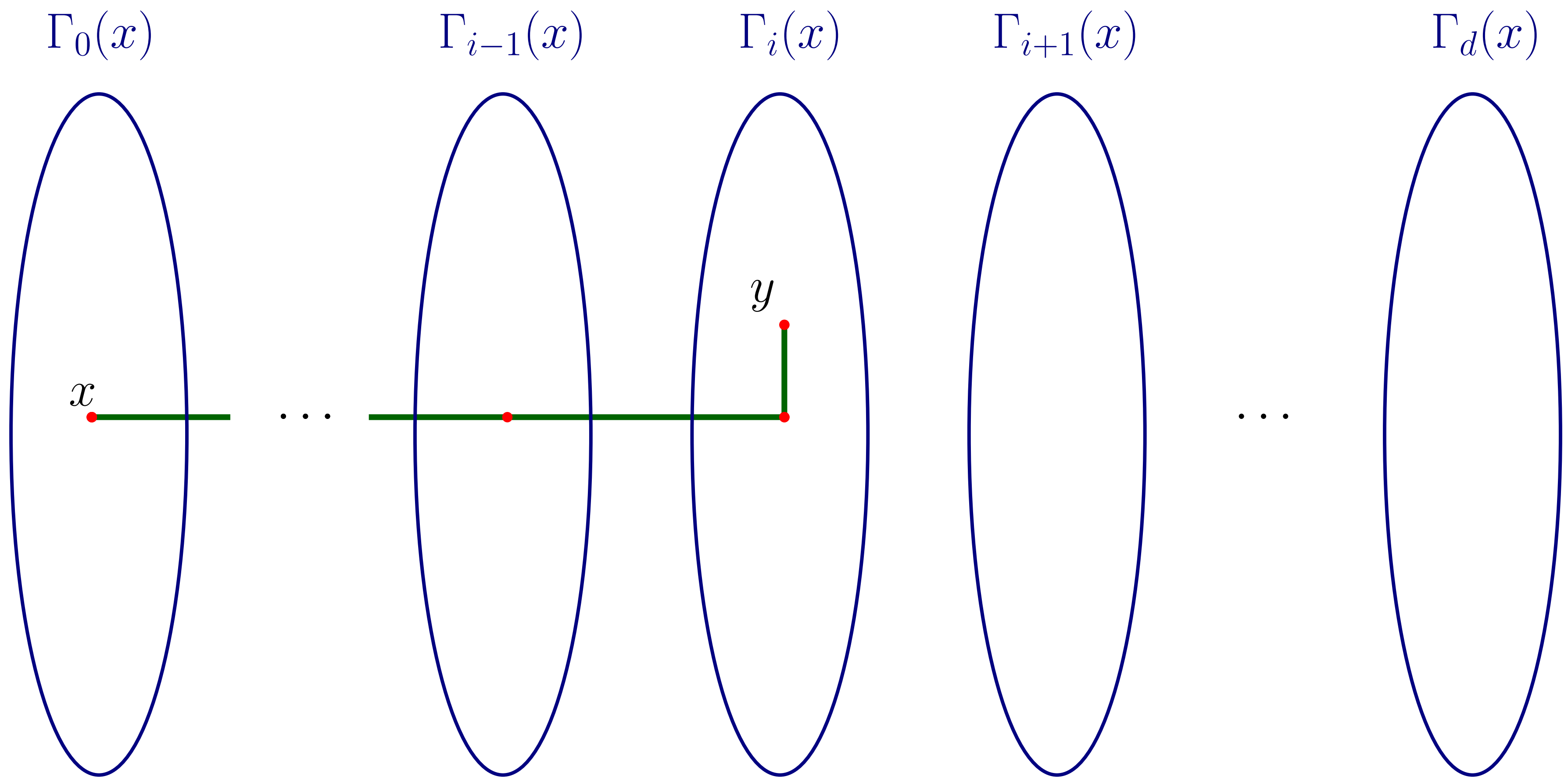
A yz-walk in Γ of shape $\ell r^2 \ell f r$ with respect to x.

Walks of a certain shape



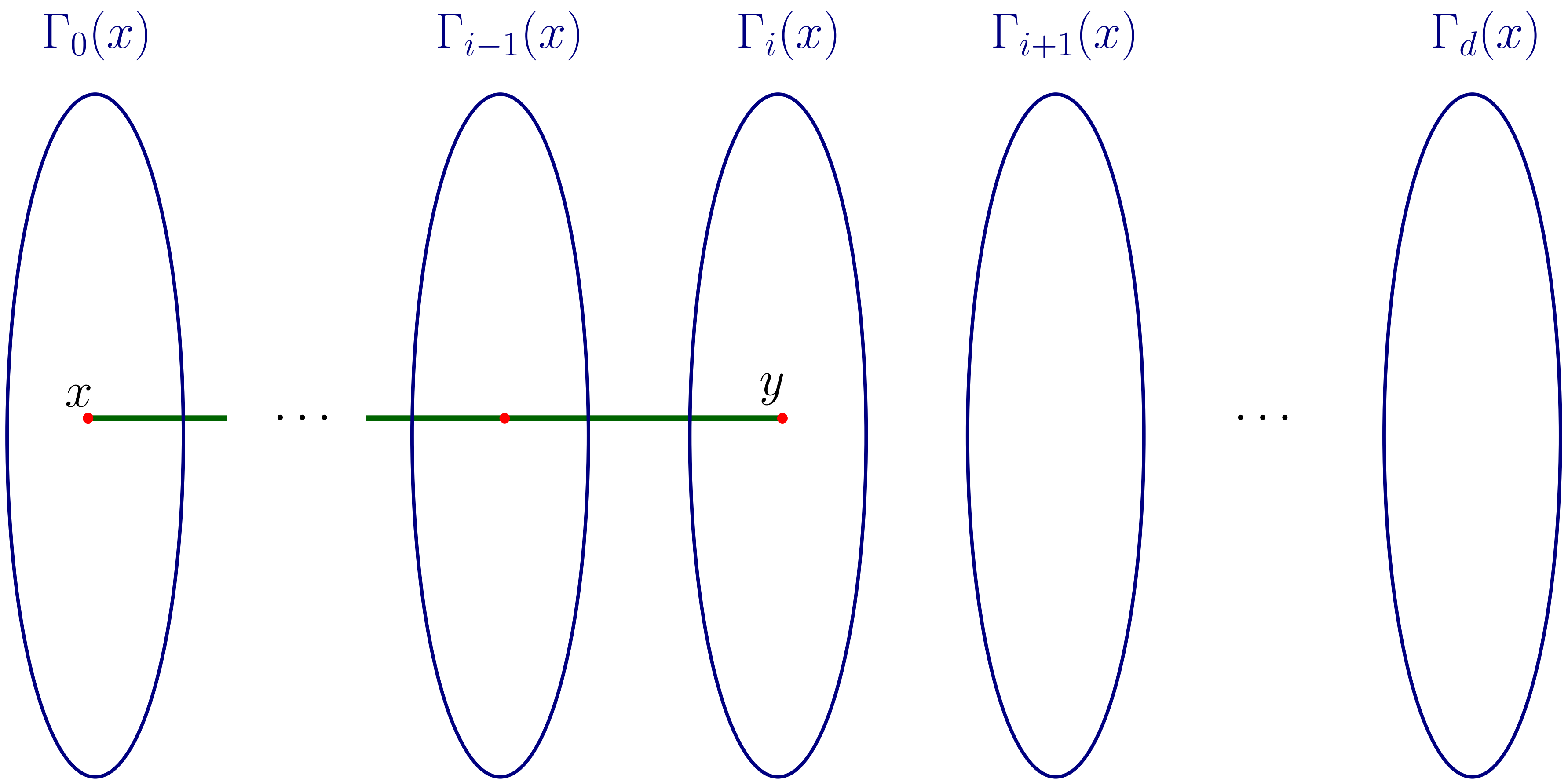
For an integer i $(0 \le i \le d)$ and $y \in \Gamma_i(x)$, let $r^{i+1}\ell(y)$ denote the number of xy-walks in Γ of shape $r^{i+1}\ell$ with respect to x.

Walks of a certain shape



For an integer i ($0 \leq i \leq d$) and $y \in \Gamma_i(x)$, let $r^i f(y)$ denote the number of xy-walks in Γ of shape $r^i f$ with respect to x.

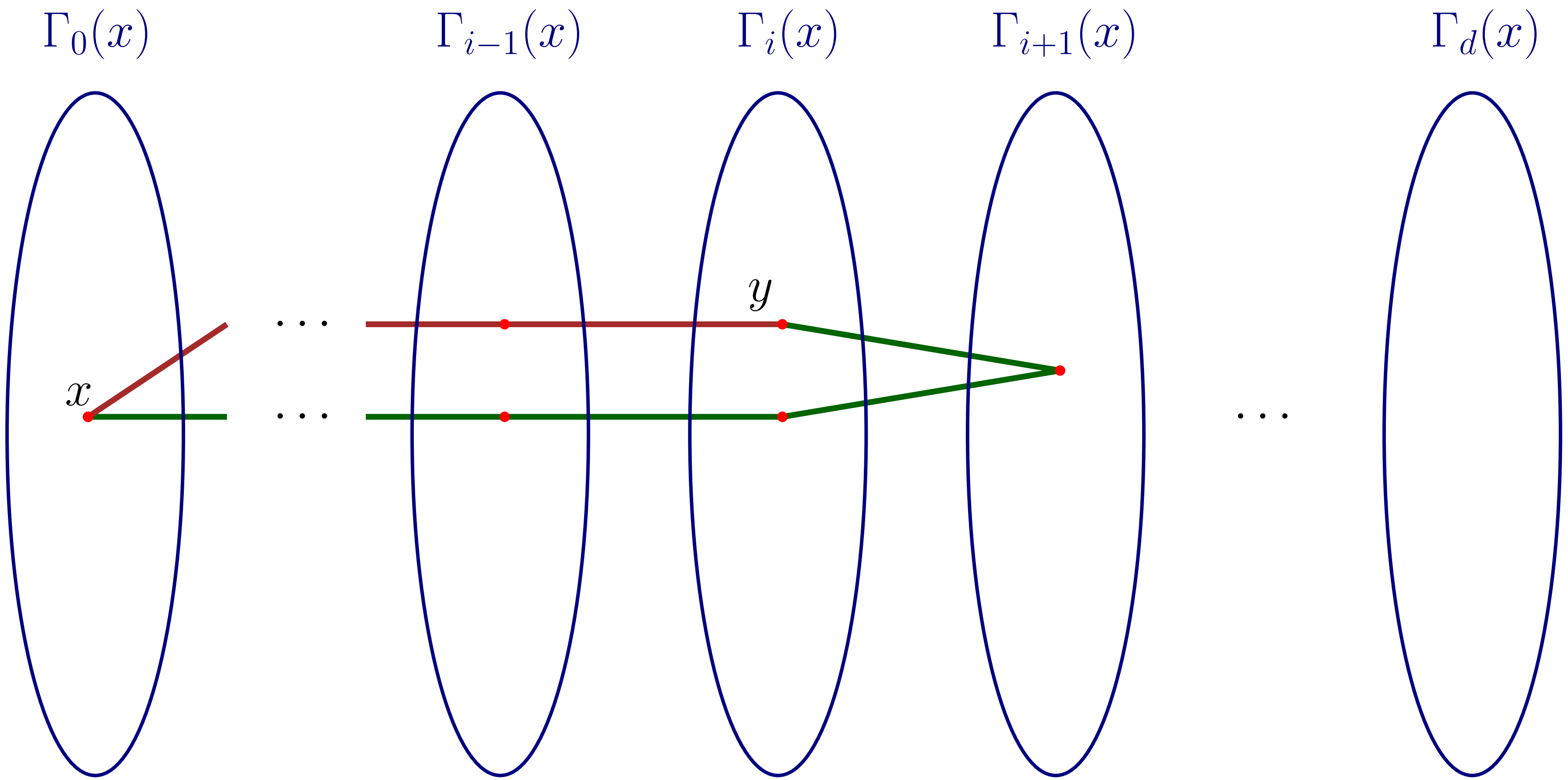
Walks of a certain shape



For an integer i ($0 \leq i \leq d$) and $y \in \Gamma_i(x)$, let $r^i(y)$ denote the number of xy-walks in Γ of shape r^i with respect to x.

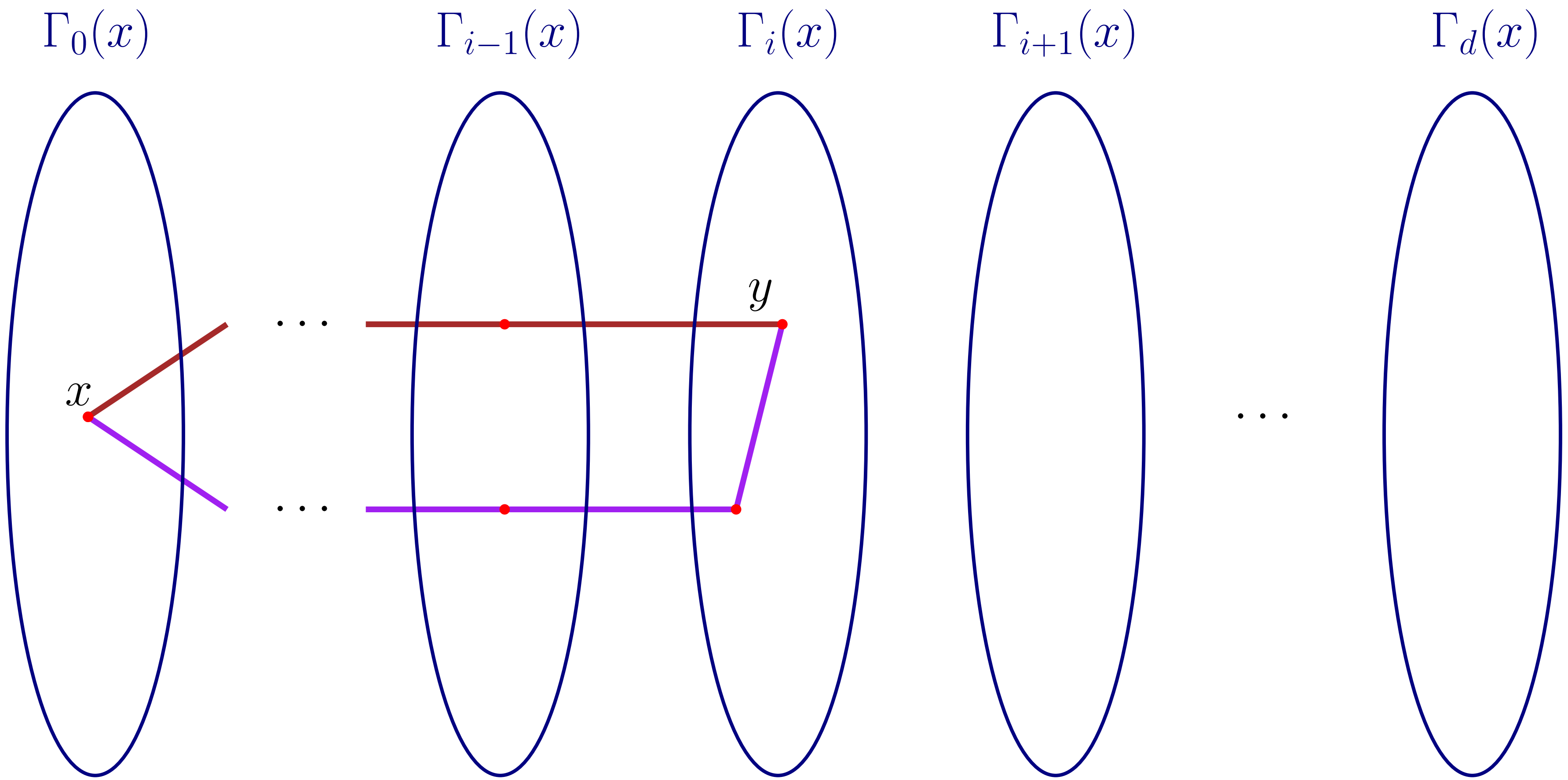

idea

A combinatorial condition



For every integer i $(0 \leq i \leq d)$ there exist scalars α_i, such that for every $y \in \Gamma_i(x)$ we have $r^{i+1}\ell(y) = \alpha_i\ r^i(y)$.

A combinatorial condition



For every integer i $(0 \leq i \leq d)$ there exist scalars β_i, such that for every $y \in \Gamma_i(x)$ we have $r^i f(y) = \beta_i \, r^i(y)$.

The result we wanted!

Theorem 1 (Fernández, Miklavič, 2022)

Let $\Gamma = (X, \mathcal{R})$ be a graph. Fix $x \in X$ and let $d = \epsilon(x)$. Consider the Terwilliger Algebra T of Γ with respect to x. The following (1)–(3) are equivalent:

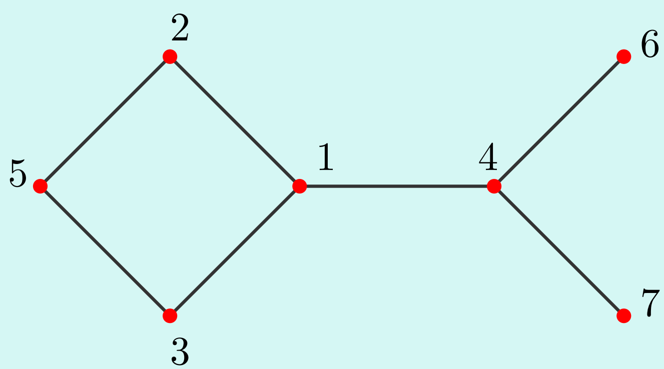
1. *$T\hat{x}$ is thin.*
2. *Γ is pseudo distance-regular around x.*
3. *For every integer i $(0 \le i \le d)$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ the following hold:*

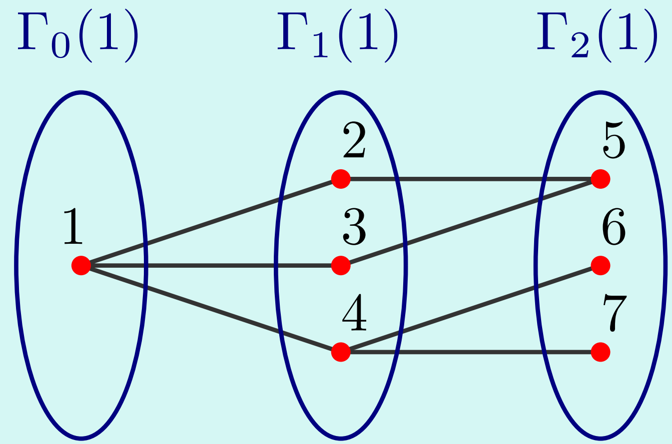
$$r^{i+1}\ell(y) = \alpha_i \, r^i(y), \qquad r^i f(y) = \beta_i \, r^i(y).$$

In particular, if the above equivalent conditions (1)–(3) hold, then the set $\{R^i\hat{x} \mid 0 \le i \le d\}$ is a basis of $T\hat{x}$.

Example

Let $\Gamma = (X, \mathscr{R})$ be the simple graph with $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathscr{R} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,5\}, \{3,5\}, \{4,6\}, \{4,7\}\}$.





The trivial module $\widehat{T1}$ is thin.

Remark

The combinatorial condition

$$r^{i+1}\ell(y) = \alpha_i \; r^i(y), \text{ for every } y \in \Gamma_i(x) \; (0 \leq i \leq 2),$$

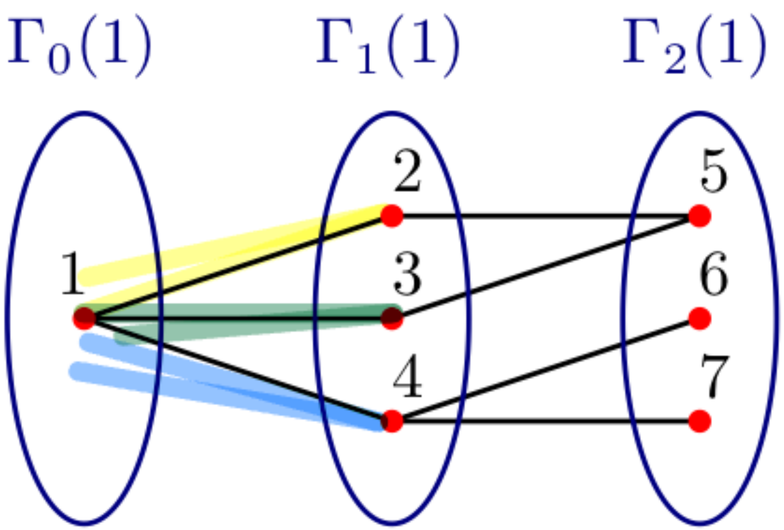
holds with $\alpha_0 = 3$, $\alpha_1 = 2$ and $\alpha_3 = 0$.
Therefore, the trivial T-module $\widehat{T1}$ is thin.

$r^{i+1}\ell(y) = \alpha_i \, r^i(y)$, for every $y \in \Gamma_i(x)$ $(0 \le i \le 2)$

FOR $i=0$ WE NEED TO CHECK THAT

$$r\ell(x) = \alpha_0 \cdot r^0(x)$$

Notice [1, 2, 1], [1, 3, 1], [1, 4, 1] are all the walks of the shape $r\ell$ with respect to 1. So, for $i = 0$ we observe $r^0(x) = 1$ and $r\ell(x) = 3$. Then, the above equation holds with $\alpha_0 = 3$.



$r^{i+1}\ell(y) = \alpha_i \, r^i(y)$, for every $y \in \Gamma_i(x)$ $(0 \le i \le 2)$

FOR $i=2$ WE NEED TO CHECK THAT

$$r^3 \ell(y) = \alpha_3 \cdot r^2(y) \quad \text{FOR ALL } y \in \Gamma_2(x)$$

WE OBSERVE THAT $r^3 \ell(y) = 0$ FOR ALL $y \in \Gamma_2(x)$.

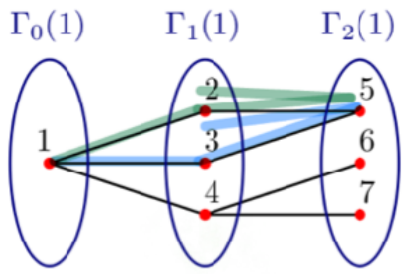
MOREOVER, $r^2(y) > 0$ FOR ALL $y \in \Gamma_2(x)$.

WE THUS HAVE THAT $\alpha_3 = 0$.

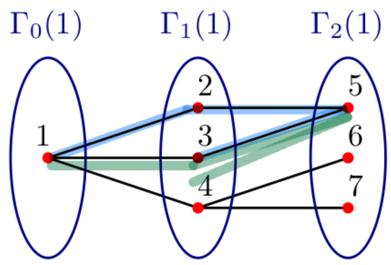
$r^{i+1}\ell(y) = \alpha_i\, r^i(y)$, for every $y \in \Gamma_i(x)$ $(0 \leq i \leq 2)$

FOR $i=1$ WE NEED TO CHECK THAT

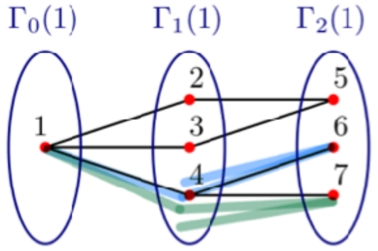
$$r^2 \ell(y) = \alpha_2 \cdot r(y) \quad \text{FOR ALL } y \in \Gamma_1(x)$$



Notice $[1,2,5,2]$, $[1,3,5,2]$ are all the $1,2$-walks of the shape $r^2\ell$ with respect to 1.



Notice $[1,2,5,3]$, $[1,3,5,3]$ are all the $1,3$-walks of the shape $r^2\ell$ with respect to 1.



Notice $[1,4,6,4]$, $[1,4,7,4]$ are all the $1,4$-walks of the shape $r^2\ell$ with respect to 1.

So, for every $y \in \Gamma_1(x)$, we have $r^2\ell(y) = 2$ and $r(y) = 1$. Then, the above equation holds with $\alpha_1 = 2$.

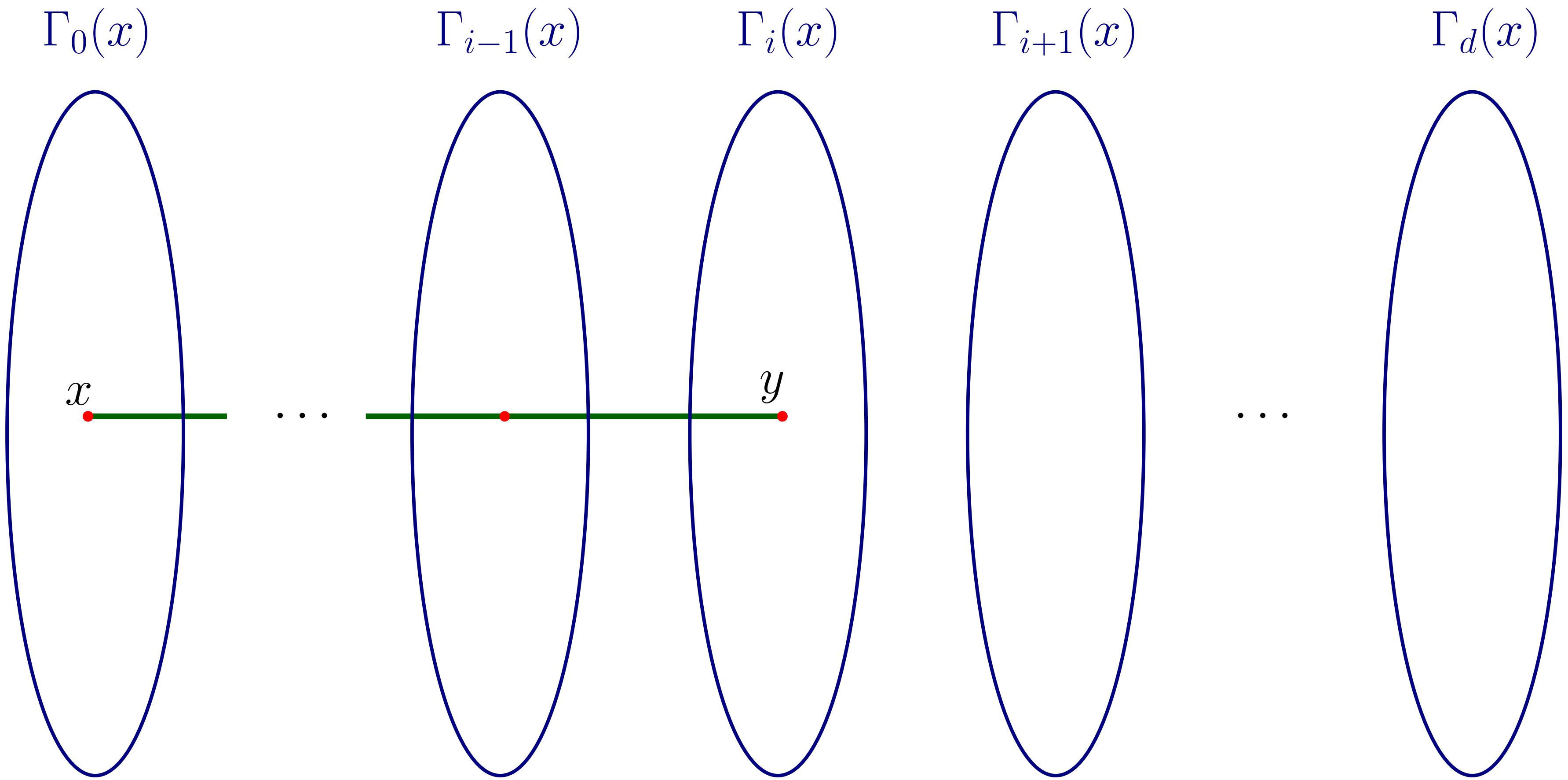
A consequence

Let $\Gamma = (X, R)$ be a graph. Fix $x \in X$ and consider the Terwilliger Algebra T of Γ with respect to x.

Corollary (Terwilliger, 1993)

If Γ is distance-regular around x then $T\hat{x}$ is thin.

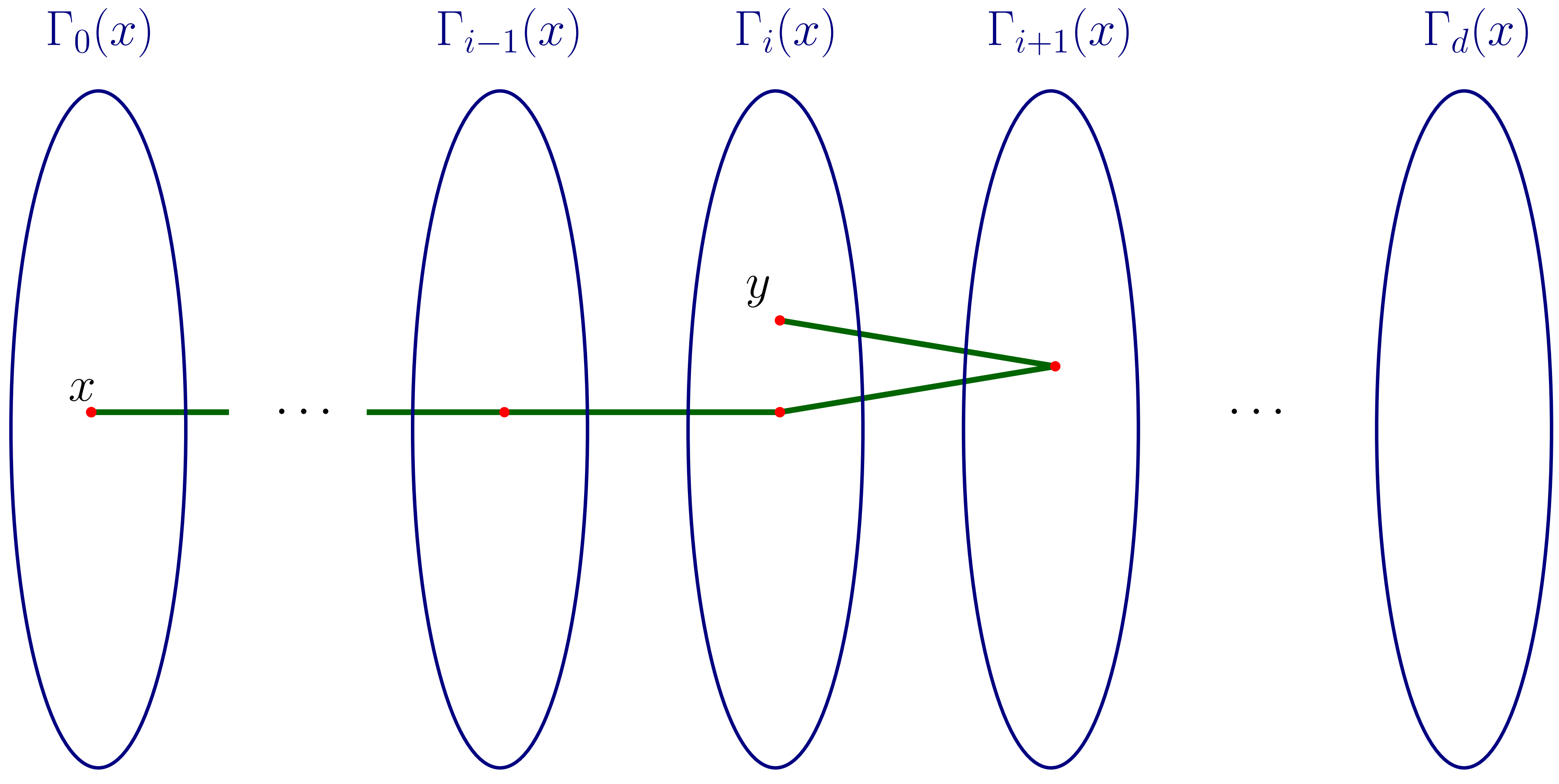
Assume that Γ is distance-regular around x.



For every integer i ($0 \leq i \leq d$) and $y \in \Gamma_i(x)$, we have

$$r^i(y) = \prod_{j=1}^{i} c_j(x)$$

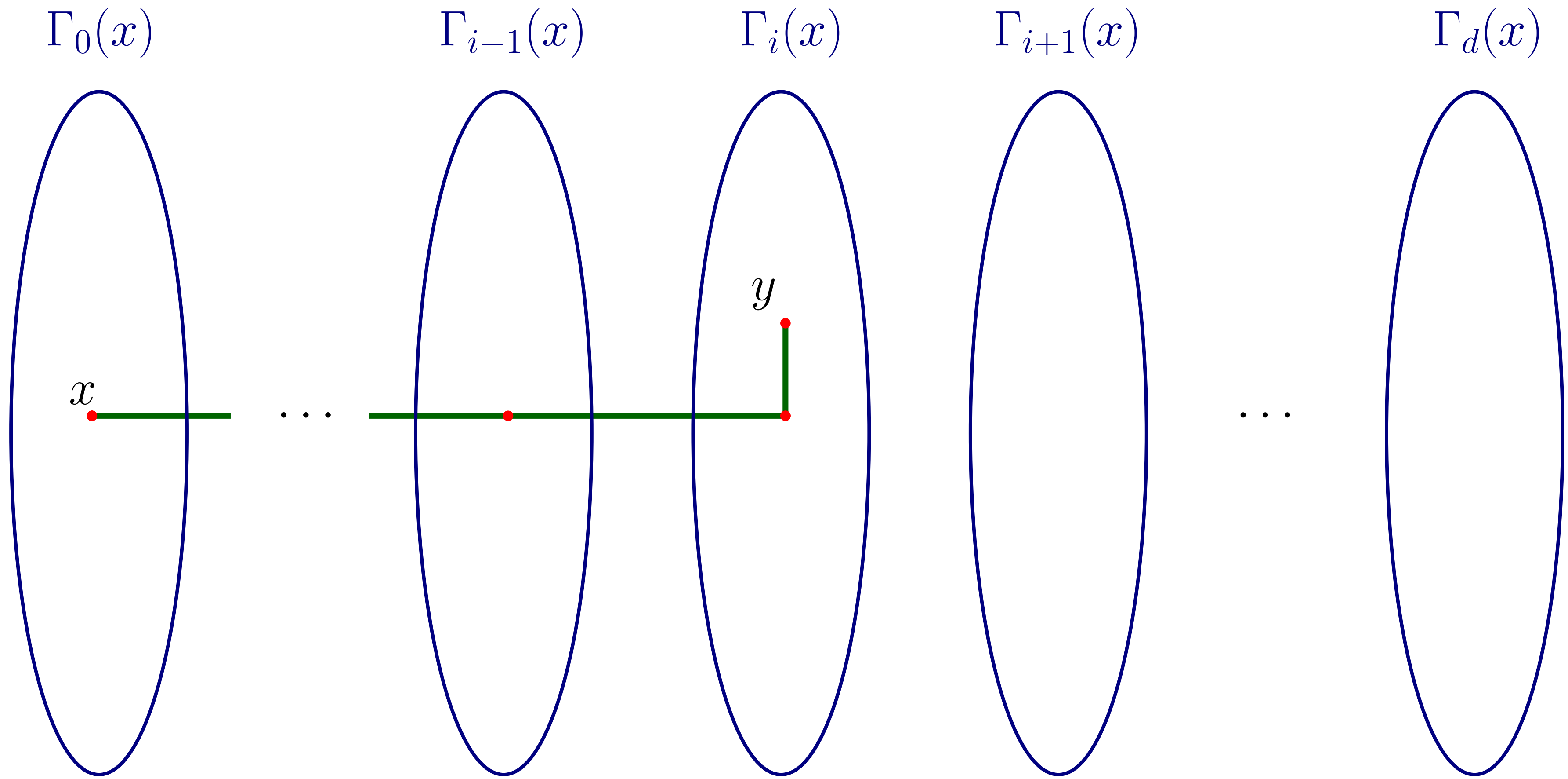
Assume that Γ is distance-regular around x.



For every integer i ($0 \leq i \leq d$) and $y \in \Gamma_i(x)$, we have

$$r^{i+1}\ell(y) = b_i(x)c_{i+1}(x)\prod_{j=1}^{i} c_j(x).$$

Assume that Γ is distance-regular around x.



For every integer i $(0 \leq i \leq d)$ and $y \in \Gamma_i(x)$, we have

$$r^i f(y) = a_i(x) \prod_{j=1}^{i} c_j(x).$$

Then, for every $y \in \Gamma_i(x)$ $(0 \leq i \leq d)$ we have:

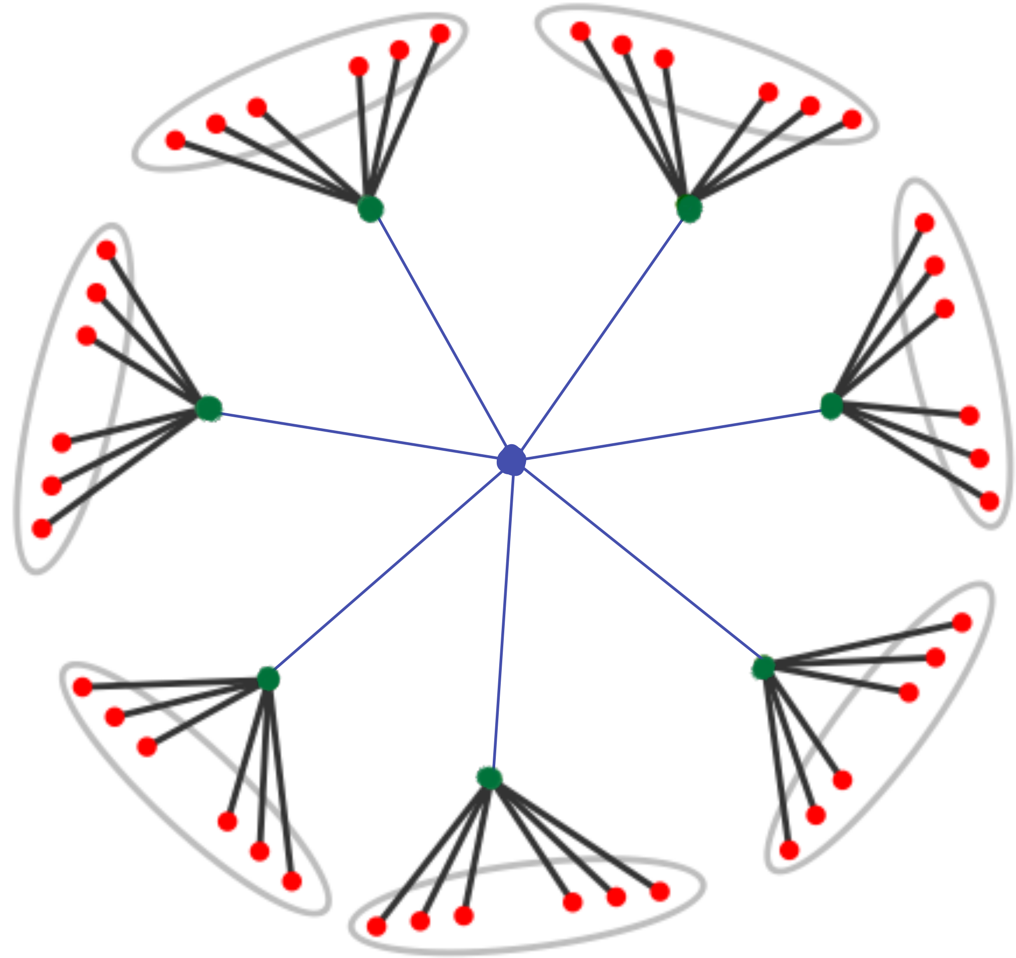
$$r^i(y) = \prod_{j=1}^{i} c_j(x)$$

$$r^{i+1}\ell(y) = b_i(x)c_{i+1}(x)\prod_{j=1}^{i} c_j(x) = b_i(x)c_{i+1}(x)r^i(y).$$

$$r^i f(y) = a_i(x)\prod_{j=1}^{i} c_j(x) = a_i(x)r^i(y).$$

Therefore, for every $y \in \Gamma_i(x)$ we have that $r^{i+1}\ell(y) = \alpha_i r^i(y)$ and $r^i f(y) = \beta_i r^i(y)$ with $\alpha_i = b_i(x)c_{i+1}(x)$ and $\beta_i = a_i(x)$. By Theorem 1, the trivial T-module $T\hat{x}$ is thin.

A construction



$$\tau \; h \; \{a_{n_k}\} \;\; y \;\; \circ \;\; \cup \; !$$

Thank You!