Extensions of Steiner loops

Mario Galici

University of Palermo

Joint work with: G. Falcone, A. Figula, G. Filippone

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Steiner triple systems

Definition

A **Steiner triple system** is a pair $(\mathcal{S}, \mathcal{T})$, where

- $\mathcal{S}$ is a set,
- $\mathcal{T}$ is a family of **triples** of $\mathcal{S}$ such that any two points of $\mathcal{S}$ are contained in exactly one triple of $\mathcal{T}$.

Loops

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A **loop** is a set $\mathcal{L}$ equipped with a binary operation $+$ with an identity element Ω such that

$$a + x = b, \tag{1}$$

$$y + a = b, \tag{2}$$

have unique solutions x and y.

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If $\mathcal{L}$ is commutative, $\mathcal{L}' \subseteq \mathcal{L}$ is **normal** if and only if

$$x + (y + \mathcal{L}') = (x + y) + \mathcal{L}', \quad \forall\, x, y \in \mathcal{L}. \tag{3}$$

Steiner loops of projective type

Steiner loops of projective type

Definition

Let $\mathcal{L}_{\mathcal{S}} := \mathcal{S} \cup \{\Omega\}$.
Defining:

$$x + \Omega = \Omega + x = x, \tag{4}$$

$$x + x = \Omega, \tag{5}$$

$$x + y = z \iff \{x, y, z\} \text{ is a triple}, \tag{6}$$

$\mathcal{L}_{\mathcal{S}}$ is a loop with identity Ω called a **Steiner loop (of projective type)**.

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$\mathcal{L}_{\mathcal{S}}$ is a group $\iff$ $\mathcal{S}$ is $\mathrm{PG}(d, 2)$.

Properties

$\mathcal{L}_{\mathcal{S}}$ is a commutative loop of exponent 2 with the **totally simmetric property:**

$$(x + y) + y = x \quad \forall\, x, y \in \mathcal{L}_{\mathcal{S}}. \tag{7}$$

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Theorem

- *Suloops of* $\mathcal{L}_{\mathcal{S}}$ *are Steiner loops.*
- *Quotients* $\mathcal{L}_{\mathcal{S}}/\mathcal{L}_{\mathcal{N}}$ *are Steiner loops* $\mathcal{L}_{\mathcal{Q}}$*.*

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- *Quotients* $\mathcal{L}_\mathcal{S}/\mathcal{L}_\mathcal{N}$ *are Steiner loops* $\mathcal{L}_\mathcal{Q}$.

We say that $\mathcal{N}$ is a normal subsystem and $\mathcal{Q}$ is the corresponding quotient system.

Steiner operators

Steiner operators

Definition

Let $\mathcal{L}_{\mathcal{N}} = \mathcal{N} \cup \{\Omega'\}$ and $\mathcal{L}_{\mathcal{Q}} = \mathcal{Q} \cup \{\bar{\Omega}\}$.
An operator

$$\begin{aligned} \Phi : \mathcal{L}_{\mathcal{Q}} \times \mathcal{L}_{\mathcal{Q}} &\longrightarrow \mathrm{Sq}(\mathcal{L}_{\mathcal{N}}) \\ (P, Q) &\longmapsto \Phi_{P,Q} \end{aligned}$$

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iii) $\Phi_{Q,P}$ is the transpose of $\Phi_{P,Q}$;

iv) $\Phi_{P,P+Q}(x, \Phi_{P,Q}(x, y)) = y$

for all $P, Q \in \mathcal{L}_{\mathcal{Q}}$, $x, y \in \mathcal{L}_{\mathcal{N}}$.

Steiner operators

$$
\begin{array}{c|ccc}
\mathcal{L}_Q & \cdots & Q & \cdots \\
\hline
\vdots & & \vdots & \\
P & \cdots & P+Q & \cdots \\
\vdots & & \vdots &
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c|ccc}
 & \cdots & \overbrace{\qquad}^{Q} & \cdots \\
\hline
\vdots & & \vdots & \\
P\left\{\vphantom{\boxed{\Phi_{P,Q}}}\right. & \cdots & \boxed{\Phi_{P,Q}} & \cdots \\
\vdots & & \vdots &
\end{array}
$$

Theorem

If we define on $\mathcal{L}_{\mathcal{Q}} \times \mathcal{L}_{\mathcal{N}}$ *the operation*

$$(P, x) + (Q, y) = \Big(P + Q,\ \Phi_{P,Q}(x, y)\Big), \tag{8}$$

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then we obtain a Steiner loop of projective type $\mathcal{L}_{\mathcal{S}}$ *such that*

$$\Omega' \longrightarrow \mathcal{L}_{\mathcal{N}} \longrightarrow \mathcal{L}_{\mathcal{S}} \longrightarrow \mathcal{L}_{\mathcal{Q}} \longrightarrow \bar{\Omega} \tag{9}$$

is short and exact.

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is short and exact.

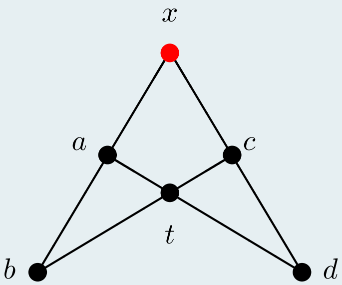
Conversely, any Steiner loop $\mathcal{L}_{\mathcal{S}}$ of projective type having a normal subloop $\mathcal{L}_{\mathcal{N}}$ and a factor loop $\mathcal{L}_{\mathcal{Q}} = \mathcal{L}_{\mathcal{S}}/\mathcal{L}_{\mathcal{N}}$, is isomorphic, for some given Steiner operator Φ, to the above one.

Veblen points

Veblen points

Definition

A point $x \in \mathcal{S}$ is a **Veblen** point if whenever $\{x, a, b\}$, $\{x, c, d\}$, $\{t, a, c\}$ are triples in $\mathcal{S}$, also $\{t, b, d\}$ is a triple in $\mathcal{S}$.



Center and Veblen points

Theorem

An element $x \neq \Omega$ *is a Veblen point* $\iff x \in \mathcal{Z} \iff \{\Omega, x\} \trianglelefteq \mathcal{L}_{\mathcal{S}}$.

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The **center** (in our case) is defined as the normal subgroup

$$\mathcal{Z} = \{z \in \mathcal{L} \mid x + (y + z) = z + (x + y) = y + (z + x),\ \forall\, x, y \in \mathcal{L}\}\,.$$

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Corollary

The set of all the Veblen points of $\mathcal{S}$ *forms a subsystem of* $\mathcal{S}$ *that is a* PG *over* GF(2).

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The operation

$$(\tau, t) \oplus (\sigma, s) = (\tau + \sigma, f(\tau, \sigma) + t^{\mathrm{T}(\sigma)} + s) \tag{10}$$

on $K \times N$ defines a **loop** $L = L(\mathrm{T}, f)$ called *Schreier extension* of N by K, such that $\bar{N} = \{(\bar{\Omega}, n) \mid n \in N\} \simeq N$ is a normal subloop and $L/\bar{N} \simeq K$.

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Schreier Extensions of Steiner loops

Remark

If $\mathcal{L}_{\mathcal{S}}$ is a Schreier extension of $\mathcal{L}_{\mathcal{N}}$ by $\mathcal{L}_{\mathcal{Q}}$, then:

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- f is **symmetric**;
- f is **constant on the triples** of $\mathcal{S}$, that is:

$$\text{if } \{P, Q, R\} \text{ is a triple } \implies f(P, Q) = f(P, R) = f(Q, R).$$

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$$\text{if } \{P, Q, R\} \text{ is a triple } \implies f(P,Q) = f(P,R) = f(Q,R).$$

The operation simply becomes

$$(P,x) + (Q,y) = (P+Q, x+y+f(P,Q)) \tag{11}$$

Schreier extensions

The function f is called a **factor system**.

The set of all Schreier extensions of $\mathcal{L}_{\mathcal{N}}$ by $\mathcal{L}_{\mathcal{Q}}$ is a group denoted by

$$\mathrm{Ext}_{\mathrm{S}}(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}}).$$

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$$|\mathrm{Ext}_{\mathrm{S}}(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}})| = |\mathcal{L}_{\mathcal{N}}|^{b} = 2^{tb},$$

where b is the number of blocks of $\mathcal{Q}$.

STS with Veblen points

Theorem

There exists a STS(v) *with (at least)* $2^c - 1$ *Veblen points if, and only if,* $\frac{v+1}{2^c} \equiv 2, 4 \pmod 6$.

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List of the first 100 order of STS's which **cannot** have Veblen points:

9, 13, 21, 25, 33, 37, 45, 49, 57, 61, 69, 73, 81, 85, 93, 97, 105, 109, 117, 121, 129, 133, 141, 145, 153, 157, 165, 169, 177, 181, 189, 193, 201, 205, 213, 217, 225, 229, 237, 241, 249, 253, 261, 265, 273, 277, 285, 289, 297, 301, 309, 313, 321, 325, 333, 337, 345, 349, 357, 361, 369, 373, 381, 385, 393, 397, 405, 409, 417, 421, 429, 433, 441, 445, 453, 457, 465, 469, 477, 481, 489, 493, 501, 505, 513, 517, 525, 529, 537, 541, 549, 553, 561, 565, 573, 577, 585, 589, 597, 601.

STS with Veblen points

"Small" cases

- The only STS(15) with Veblen points are PG(3, 2) and # 2.*

* *Handbook of Combinatorial Designs*, C. J. Colbourn, J. H. Dinitz

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- And for STS of order 19, 27 31...?

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Veblen points

Theorem

Any Schreier extension $\mathcal{L}_{\mathcal{S}}$ of index at most 4 is a group.

Corollary

If a Steiner triple system $\mathcal{S}$ with cardinality $|\mathcal{S}| < 2^d - 1$, $d > 0$, contains at least 2^{d-4} Veblen points, then it is a projective geometry.

Equivalent and isomorphic Schreier extensions

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Two Schreier extensions $\mathcal{L}_{\mathcal{S}}$ and $\mathcal{L}_{\mathcal{S}'}$ of $\mathcal{L}_{\mathcal{N}}$ by $\mathcal{L}_{\mathcal{Q}}$ are said:

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Two Schreier extensions $\mathcal{L}_{\mathcal{S}}$ and $\mathcal{L}_{\mathcal{S}'}$ of $\mathcal{L}_{\mathcal{N}}$ by $\mathcal{L}_{\mathcal{Q}}$ are said:

- **equivalent** if there is an isomorphism $\mathcal{L}_{\mathcal{S}} \to \mathcal{L}_{\mathcal{S}'}$ which induces the identity both on $\mathcal{L}_{\mathcal{N}}$ and $\mathcal{L}_{\mathcal{Q}}$.

$$f_1 \equiv f_2. \tag{12}$$

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- **isomorphic** if there is an isomorphism $\varphi : \mathcal{L}_{\mathcal{S}} \to \mathcal{L}_{\mathcal{S}'}$ such that

$$\varphi(\mathcal{L}_{\mathcal{N}}) = \mathcal{L}_{\mathcal{N}} \text{ and } \varphi(\mathcal{L}_{\mathcal{Q}}) = \mathcal{L}_{\mathcal{Q}}.$$

Equivalent Schreier extensions

Remark

$f_1, f_2 \in \mathrm{Ext}_{\mathrm{S}}(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}})$ are equivalent $\iff f_1 - f_2 = \delta^1\varphi$, for a suitable function φ.
The equivalence is given by

$$(P, x) \mapsto (P, x + \varphi(P)). \tag{13}$$

Remark

The number of non-equivalent Schreier extensions is

$$\frac{2^{tb}}{|\,\mathrm{B}^2(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}})|},$$

where

$$\mathrm{B}^2(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}}) := \{\delta^1\varphi \in \mathrm{Ext}_{\mathrm{S}}(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}}) \mid \varphi\colon \mathcal{L}_{\mathcal{Q}} \to \mathcal{L}_{\mathcal{N}}\}.$$

Action of $\mathrm{Aut}(\mathcal{L}_{\mathcal{N}})$ and $\mathrm{Aut}(\mathcal{L}_{\mathcal{Q}})$ on $\mathrm{Ext}_{\mathrm{S}}(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}})$.

Proposition

$f_1, f_2 \in \mathrm{Ext}_{\mathrm{S}}(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}})$ *are isomorphic* $\iff$ $\alpha f_1 = f_2 \beta$ *(up to an equivalence), for some* $\alpha \in \mathrm{Aut}(\mathcal{L}_{\mathcal{N}})$, $\beta \in \mathrm{Aut}(\mathcal{L}_{\mathcal{Q}})$.

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Remark

We have an action of the group $\mathrm{Aut}(\mathcal{L}_{\mathcal{N}}) \times \mathrm{Aut}(\mathcal{L}_{\mathcal{Q}})$ on the set of non-equivalent extensions

$$(\alpha, \beta)(f) = \alpha^{-1} f \beta, \tag{14}$$

whose orbits are the isomorphism classes of all the factor systems.

Veblen points in a STS(19)

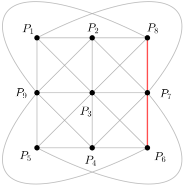
Theorem

Among the 11084874829 non-isomotphic STS(19)*, there are only 3 Steiner triple systems with one Veblen point.*

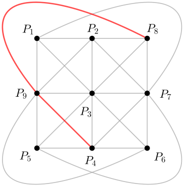
Idea of proof

- $\mathcal{S}$ be a STS(19) with a Veblen point;
- $\Omega' \longrightarrow \mathcal{L}_{\mathcal{N}} \longrightarrow \mathcal{L}_{\mathcal{S}} \longrightarrow \mathcal{L}_{\mathcal{Q}} \longrightarrow \bar{\Omega}$
  with $|\mathcal{L}_{\mathcal{N}}| = 2$ and $\mathcal{Q} = \mathrm{STS}(9)$.
- $|\mathrm{Ext}_S(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{Q}})| = 2^{12} = 4096$
- The number of non-equivalent extension is $\frac{|\mathrm{Ext}_S(\mathcal{L}_{\mathcal{N}},\mathcal{L}_{\mathcal{Q}})|}{|\mathrm{B}^2(\mathcal{L}_{\mathcal{N}},\mathcal{L}_{\mathcal{Q}})|} = 8$.
- We computed the 8 non-equivalent factor system and denote them with $f_0, f_1, \ldots, f_7$, where f_0 is the trivial one.
- $\mathrm{Aut}(\mathcal{L}_{\mathcal{N}}) = \{\mathrm{id}\}$ and $|\mathrm{Aut}(\mathcal{L}_{\mathcal{Q}})| = 432$.
- We computed $\mathrm{Aut}(\mathcal{L}_{\mathcal{Q}})$ and we found out the orbits.

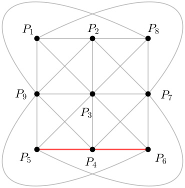
Non-trivial orbit $\{f_1, f_2, f_4, f_7\}$



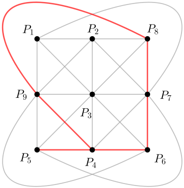
(a) f_1



(b) f_2

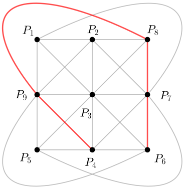


(c) f_4

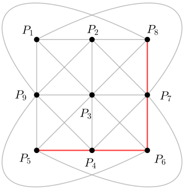


(d) f_7

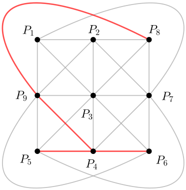
Non-trivial orbit $\{f_3, f_5, f_6\}$



(e) f_3



(f) f_5



(g) f_6

STS(27)

Theorem

There are 1736 *non-isomotphic* STS(27) *containing one Veblen point.*

- 1504: $\mathcal{Q}$ is the non-cyclic STS(13);
- 232: $\mathcal{Q}$ is the cyclic STS(13).

STS(31)

Theorem

Number of some non-isomorphic STS(31) *with one Veblen point and corresponding quotient* STS $\mathcal{Q}$*:*

$\mathcal{Q}$	*Count*
PG$(3,2)$	1240
STS(15)#2	48080
STS(15)#3	47744
STS(15)#7	16520
STS(15)#61	99952
STS(15)#80	17888

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STS(15)#80	17888

Theorem

There are only 3 *non isomorphic* STS(31) *containing precisely* 3 *Veblen points.*

Thank you
for your attention

Hvala