On strongly regular graphs decomposable into a divisible design graph and a Hoffman coclique

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(joint work with Vladislav Kabanov)

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Motivation

(1) In 2022, Vladislav Kabanov described a construction of strongly regular graphs (SRGs) based on:

- a **divisible design graph** Δ with specific parameters,
- a coclique C,
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- Thus, $(1) + (2)$ gives a *prolific* construction of SRGs
  - in which C turns out to be a Hoffman coclique.

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  (1) we show that the parameters of Δ should coincide (to some extent) with those required in the construction by Kabanov;
  (2) however, these parameters do not determine the structure of Δ, which allows us to further slightly generalize the construction of SRGs.
- To put these into a general context, I will start off with an overview of *prolific* constructions of SRGs.

Affine designs: a key ingredient in all recipes

An **affine design** $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with parameters q and r is a 2-design:

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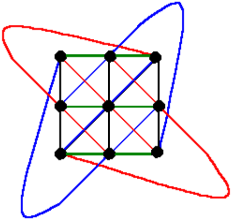
- All parameters of $\mathcal{D}$ are expressed in terms of q and r:

v:	$q^2 r$	the number of points
b:	$q^3 e + q^2 + q$	the number of blocks
k:	qr	the size of a block
λ:	$qe + 1$	the number of blocks on any two points
m:	$q^2 e + q + 1$	the number of parallel classes

  where $e = \frac{r-1}{q-1}$ is an integer.

Affine designs: examples

- an affine plane of order q (with $r = 1$):
  - $\mathcal{P}$: points,
  - $\mathcal{B}$: lines.



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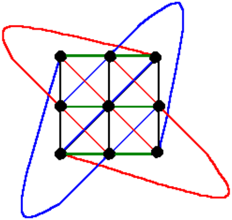

- the d-dimensional affine space over $\mathbb{F}_q$ (with $r = q^{d-2}$):
  - $\mathcal{P}$: points,
  - $\mathcal{B}$: hyperplanes.
- Hadamard 3-designs (with $q = 2$).

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- and a collection of V affine designs $\mathcal{D}_1, \ldots, \mathcal{D}_V$ with the same parameters q and r (not necessarily isomorphic):
  - V, B, R, K and q, r are related; in particular, $R =$ the number of parallel classes in $\mathcal{D}_i$,
  - let $\underbrace{\mathcal{L}_i, \mathcal{L}_i, \mathcal{L}_i, \ldots}_{R \text{ colors}}$ be all parallel classes of $\mathcal{D}_i = (\mathcal{P}_i, \underbrace{\mathcal{L}_i \cup \mathcal{L}_i \cup \mathcal{L}_i \cup \ldots}_{\mathcal{B}_i})$.

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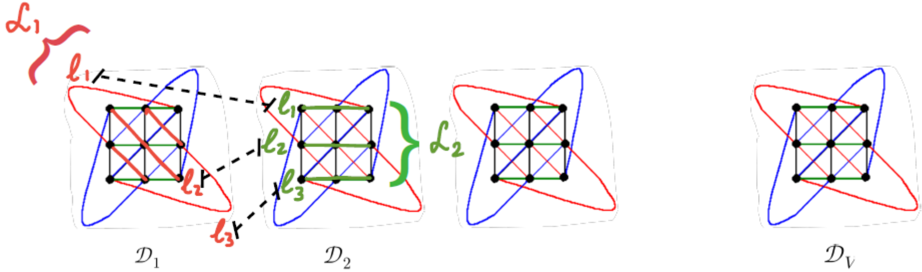
- Identify the point set of Σ with $\{1, 2, \ldots, V\}$.
- Identify the blocks of Σ with collections of parallel classes in such a way that all blocks containing a point i exhaust all R colors $\mathcal{L}_i, \mathcal{L}_i, \mathcal{L}_i, \ldots$:

$$\text{e.g., } R \text{ blocks containing } 1: \begin{cases} \{1,2,3,\ldots\} \mapsto \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots\} \\ \{1,4,5,\ldots\} \mapsto \{\mathcal{L}_1, \mathcal{L}_4, \mathcal{L}_5, \ldots\} \\ \{1,6,7,\ldots\} \mapsto \{\mathcal{L}_1, \mathcal{L}_6, \mathcal{L}_7, \ldots\} \\ \ldots \end{cases}$$

Wallis: construction

Now define a graph Γ on $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \mathcal{P}_V$:

- each $\mathcal{P}_i$ induces an empty graph (coclique),
- for every block, pairwisely join points on "*corresponding*" lines from different parallel classes:



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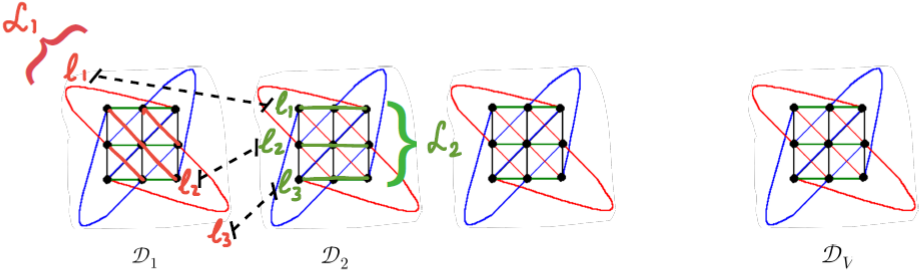
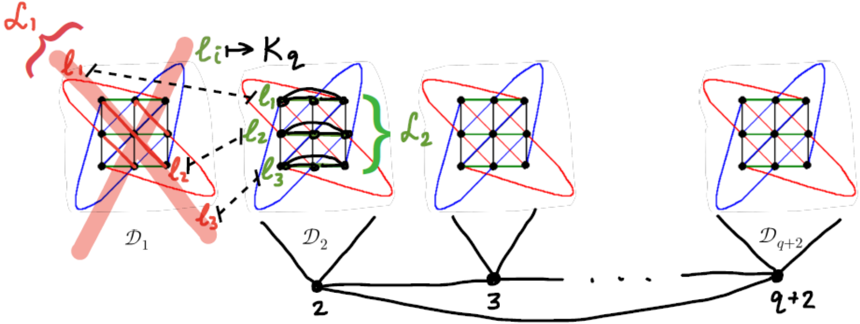


- Then Γ is an **SRG**.
- Lots of freedom, but Wallis did not estimate the number of non-isomorphic graphs.

Fon-Der-Flaass (2002)

- The paper by Wallis "went largely unnoticed" and in 2002 Fon-Der-Flaass reinvented some of Wallis' ideas.
- He discovered three more constructions; one of them is a special case of the Wallis' one.
- He also showed that it produces hyperexponentially many (as q increases) non-isomorphic SRGs with the same parameters.

Fon-Der-Flaass-2



Take Γ constructed earlier (using affine *planes* of order q and $\Sigma = \binom{q+2}{2}$):

- remove one of the affine planes,
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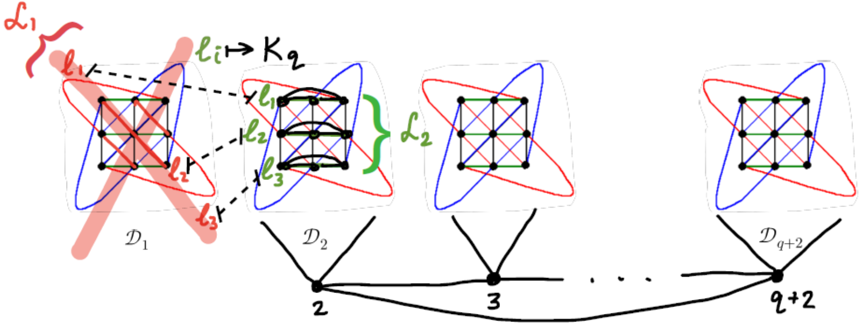

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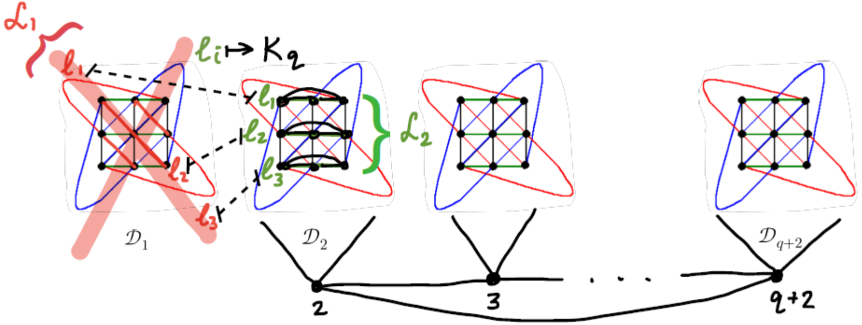

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- remove one of the affine planes,
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- turn all lines in the "free" parallel classes into q-cliques.
- finally, add a $(q+1)$-clique K and join the i-th vertex of K to all vertices of $\mathcal{P}_i$,

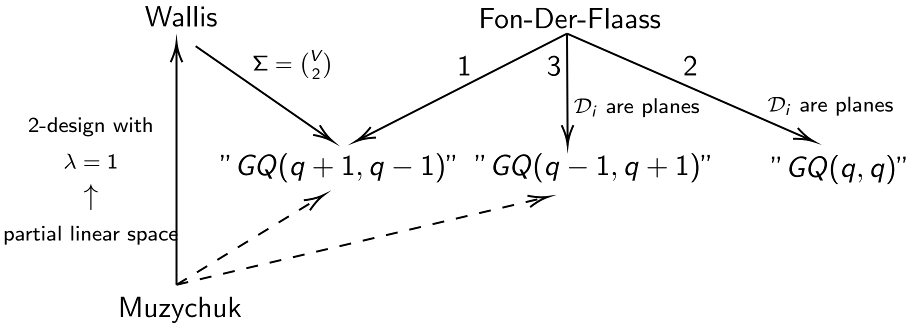
This produces an SRG with the parameters of $GQ(q,q)$.

Muzychuk (2007)

Muzychuk went further and generalized the idea of Wallis:

- instead of a Steiner 2-design Σ he takes a partial linear space whose collinearity graph is an SRG.
- he constructed at least 6 families of SRGs, including Fon-Der-Flaass-1 and Fon-Der-Flaass-3.

Big picture



Fon-Der-Flaass-2 looks odd in this picture.

DDGs

- Divisible design graph (Haemers, Kharaghani, Meulenberg, 2011):
  - a k-regular graph on $v = mn$ vertices,
  - the vertex set can be partitioned into m classes of size n, such that:

$$\#\text{common neighbors of } x, y = \begin{cases} \lambda_1, & \text{if } x, y \text{ are from the same class,} \\ \lambda_2, & \text{if } x, y \text{ are from different classes} \end{cases}$$

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$$(v, k, \lambda_1, \lambda_2; m, n)$$

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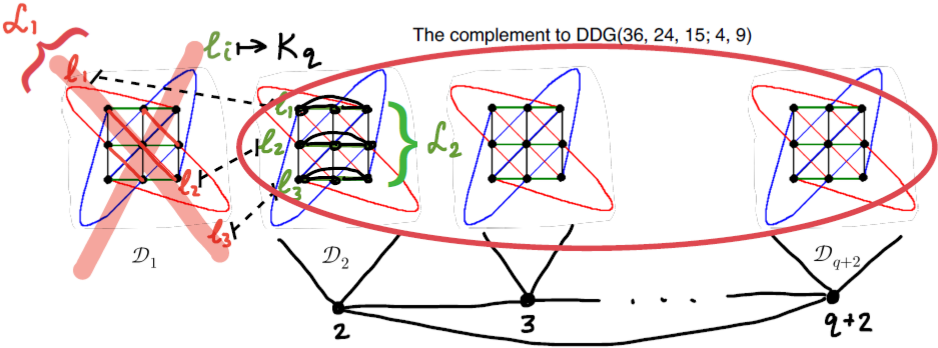
Kabanov then took a closer look at these examples and noticed that quite a few of them arise in the Wallis – Fon-Der-Flaass manner.



Prolific constructions of DDGs

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- Why was this overlooked?

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$$\left(\underbrace{\frac{q^d(q^d-1)}{q-1}}_{v}, \underbrace{q^{d-1}(q^d-1)}_{k}, \underbrace{q^{d-1}(q^d-q^{d-1}-1)}_{\lambda_1}, \underbrace{q^{d-2}(q^d-1)(q-1)}_{\lambda_2}; \underbrace{\frac{q^d-1}{q-1}}_{m}, \underbrace{n=q^d}_{n}\right)$$

DDG + coclique = SRG

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Just like in the Fon-Der-Flaass-2 construction:

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$+ \leftarrow$ symmetric design

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$||$

$$\mathrm{SRG}\left(\frac{q^{2d}-1}{q-1}, q^{2d-1}, q^{2d-2}(q-1), q^{2d-2}(q-1)\right)$$

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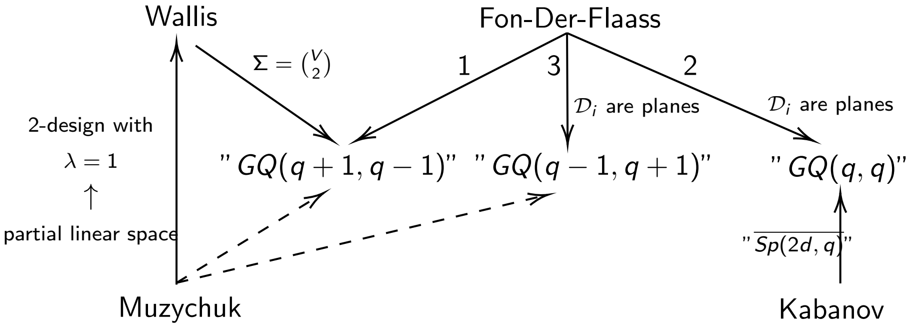
These are the parameters of the complement of $\text{Sp}(2d, q)$.

Big picture updated



Sp($2d, q$): "prolific" parameters

- Abiad, Haemers (2016): Godsil-McKay switching for $q = 2$,
- Kubota (2016): more Godsil-McKay switching for $q = 2$,
- Cossidente, Pavese (2017): "geometric" switching in generalized quadrangles,
- Ihringer (2017): "geometric" switching in polar spaces,
- Brouwer, Ihringer, Kantor (2021): "geometric" switching in symplectic polar spaces preserving the 4-vertex condition.

Our result

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What we show:

- the parameters of Γ have the following form:

$$v = (-s)\frac{n^2 - 1}{n + s}, \quad k = (-s)n, \quad \lambda = \mu = (-s)(n + s),$$

  where s is the smallest eigenvalue of Γ.

- If $-s$ is a prime power q, these are the parameters of $\overline{\mathrm{Sp}(2d, q)}$.

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- However, not all of DDGs with the required parameters arise from affine planes.
  - there are 28 SRGs $(40, 27, 18, 18)$ (Spence, 2000),
  - 27 of them decompose into “$\mathrm{DDG}(36, 24, 15; 4, 9)$ + 4-coclique”,
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  It exists if there exists a DDG$(252, 210, 174, 175; 7, 36)$.

Sketch

Suppose Γ has spectrum k^1, r^f, s^g.

- $\Gamma \setminus C$ has spectrum with 4 eigenvalues

$$(k+s)^1, r^{f-c+1}, (r+s)^{c-1}, s^{g-c}$$

- DDG $(v', k', \lambda_1, \lambda_2; m, n)$ has 5 *eigenvalues*

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  but their multiplicities are not determined in general.

Then we have two principal cases:

- either $\pm\sqrt{k'-\lambda_1} = 0$ or $\pm\sqrt{k'^2-\lambda_2 v'} = 0$:
  - in one of these two cases we obtain our parameters,
- or one of the eigenvalues in the spectrum of DDG collapses (has 0 multiplicity):
  - 8 sub-cases depending on whether $\sqrt{k'-\lambda_1} \lessgtr \sqrt{k'^2-\lambda_2 v'}$.

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- Since the complement to a DDG is not a DDG, our result does not immediately extend to the situation "DDG + a (Delsarte) clique".
- We do know some examples when this happens:
  - there are 3854 SRGs $(35, 18, 9, 9)$ (McKay, Spence, 2001).
  - 499 of them are DDG$(28, 15, 6, 8; 7, 4)$ + 7-clique.

THANK YOU

The co-author