Violator Spaces and Greedoids

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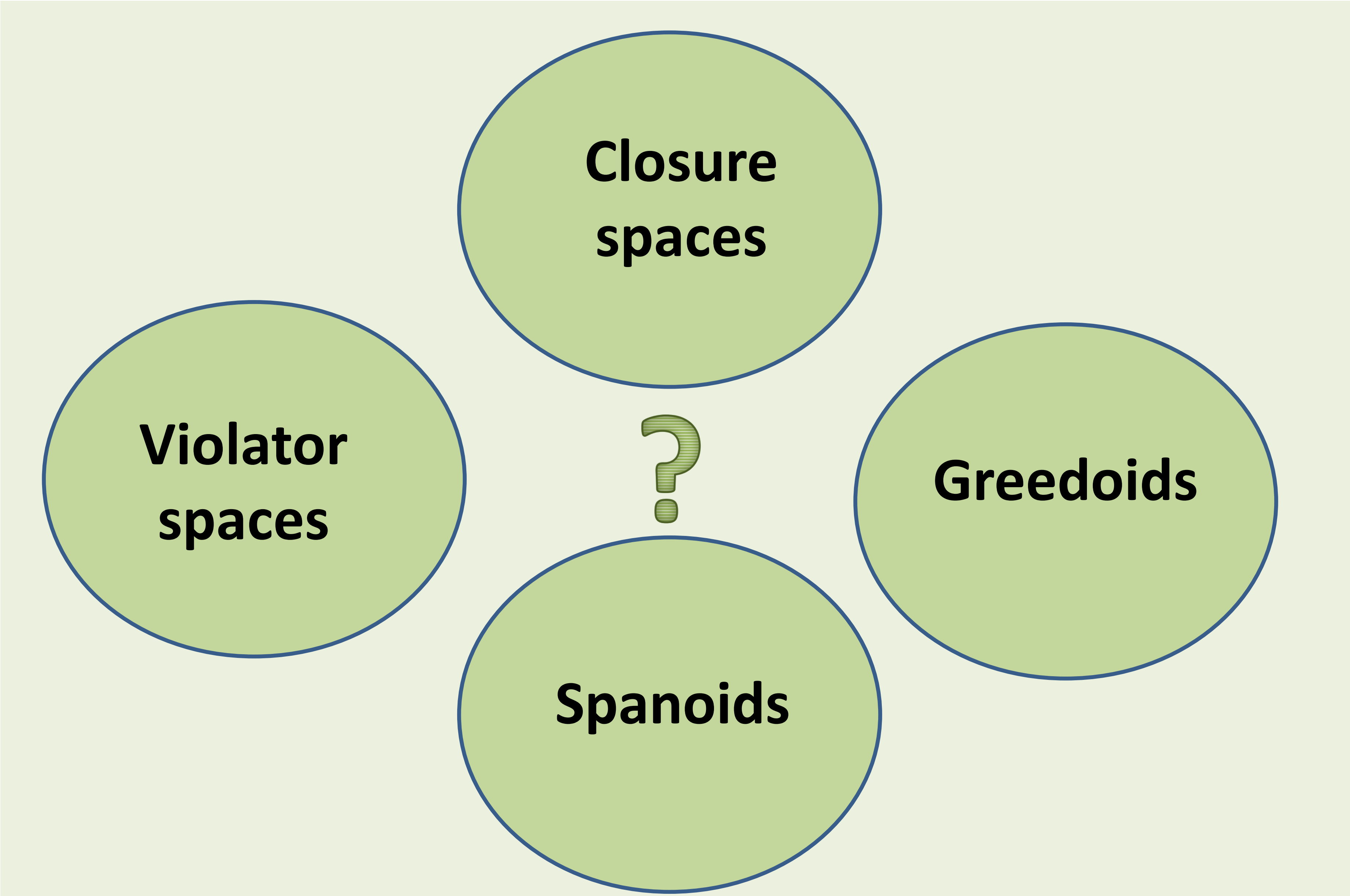
Closure spaces
Violator spaces
Greedoids
Spanoids
?

Outline

Combinatorial structures:

- violator spaces
- closure spaces
- spanoids
- greedoids

- **Set systems** or Hypergraphs: (E,F)

  E – a finite ground set and a family of feasible sets $F \subseteq 2^E$

- **Spaces** : (E,α)

  E – a ground set and an operator $\alpha : 2^E \to 2^E$

Closure spaces: Definition

$\tau : 2^E \rightarrow 2^E$ is a **closure operator** if:

- for all $X \subseteq E$, $X \subseteq \tau(X)$ (*extensivity*)
- for all $X \subseteq Y \subseteq E$, $\tau(X) \subseteq \tau(Y)$ (*isotonicity*)
- for all $X \subseteq E$, $\tau(\tau(X)) = \tau(X)$ (*idempotence*)
- A ***closure space*** is a pair (E, τ), where E is a finite set and $\tau : 2^E \rightarrow 2^E$ is a closure operator.

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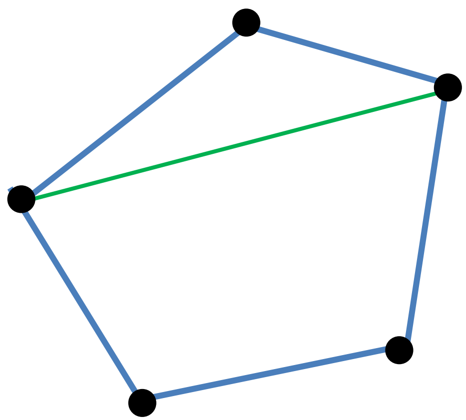
A set $A \subseteq E$ is ***closed*** if $A = \tau(A)$.

The family of closed sets K is closed under intersection:

$$X, Y \in K \rightarrow X \cap Y \subseteq \tau(X \cap Y) \subseteq \tau(X) \cap \tau(Y) = X \cap Y$$

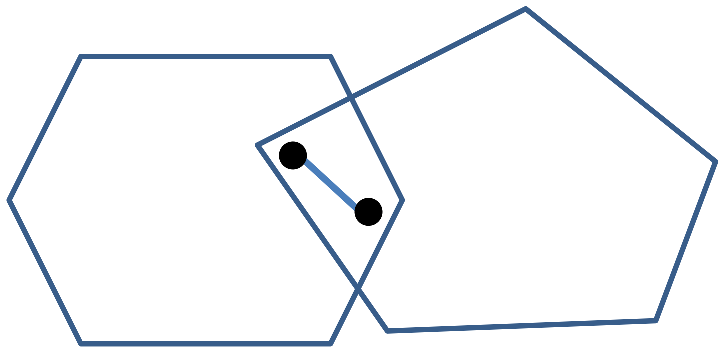
Conversely, any set system (E,K) closed under intersection is a family of closed sets of the closure operator $\tau(X) = \cap\{A \in K : X \subseteq A\}$.

Convex sets – closed sets



In a Euclidean space, a set is convex if it contains the line segment between any two of its points.

It is easy to see that the family of convex sets is closed under intersection.



In fact, the family of convex sets coincides with the family of closed sets defined by the **convex hull operator –** closure operator τ.

Violator spaces. Definition.

Violator Spaces were introduced by Matoušek et al. in 2008 as generalization of Linear Programming problems.

A ***violator space*** is a finite space $(\boldsymbol{H},\boldsymbol{V})$, where

$\boldsymbol{V} : 2^{H} \rightarrow 2^{H}$ is a mapping such that:

- for all $\boldsymbol{X} \subseteq \boldsymbol{H}$, $\boldsymbol{X} \cap \boldsymbol{V}(\boldsymbol{X}) = \varnothing$ (consistency)
- for all $\boldsymbol{X} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{H}$, s.t. $\boldsymbol{Y} \cap \boldsymbol{V}(\boldsymbol{X}) = \varnothing$, $\boldsymbol{V}(\boldsymbol{X}) = \boldsymbol{V}(\boldsymbol{Y})$ (locality)

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An Interpretation:

- ***H*** is the set of constraints
- ***V***(***X***) - the set of all constraints violating ***X***

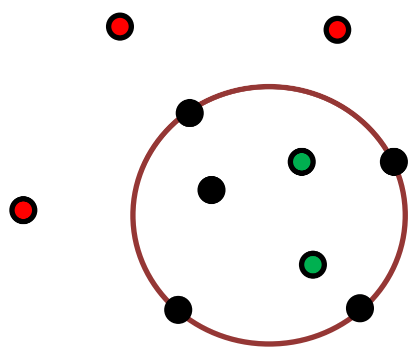
For $\boldsymbol{X \subseteq H}$, a ***basis*** of ***X*** is a minimal subset ***B*** with $\boldsymbol{V(B) = V(X)}$.

Violator spaces: Example

The smallest enclosing ball in R^2:

<u>Problem</u>: Given a set of points in $\boldsymbol{R}^2$,
find the smallest circle containing them.

- ***H*** *is* a set of points $\boldsymbol{R}^2$.
- ***V***: For $\boldsymbol{X} \subseteq \boldsymbol{H}$, a point ***p*** outside of ***X*** "violates" ***X*** if adding ***p*** to ***X*** increases the size of the smallest circle containing ***X***.



X- **black** points
Red points violate ***X***
Green (and black) point does not violate ***X***
A basis of ***X*** – any three points on the circle.

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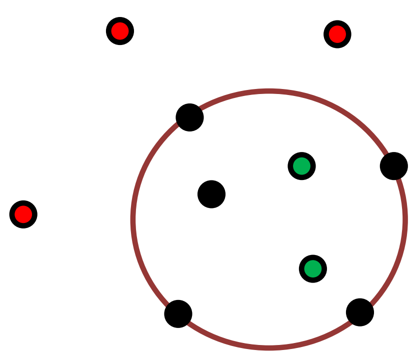
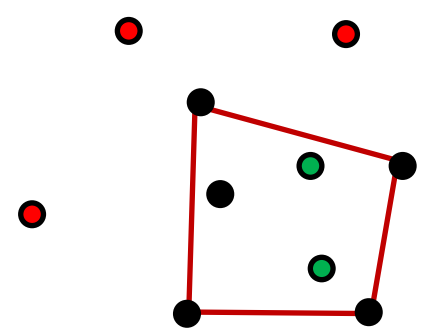


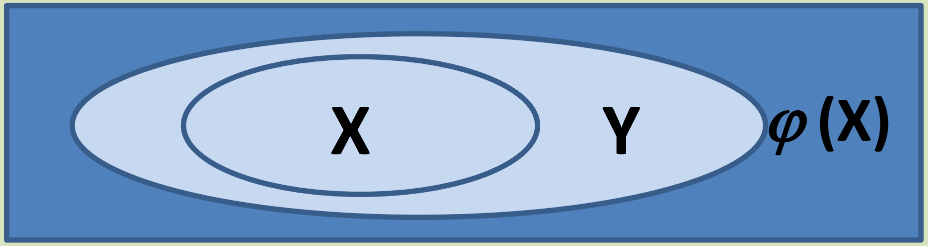
convex hull operator: a basis of $\boldsymbol{X}$ – the extreme points of the convex hull.



Violator operator

Let $\boldsymbol{V}$ be a violator mapping. Define $\boldsymbol{\varphi(X) = H - V(X)}$.

- Consistency of $\boldsymbol{V}$ is equivalent to ***extensivity*** of $\boldsymbol{\varphi}$:
  $\boldsymbol{X \cap V(X) = \varnothing \leftrightarrow X \subseteq \varphi(X)}$
- Locality of $\boldsymbol{V}$ is equivalent to "***self-convexity***":
  $\boldsymbol{X \subseteq Y \subseteq \varphi(X) \Rightarrow \varphi(X) = \varphi(Y)}$



- In what follows the pair (H, φ) , where φ is extensive and self-convex operator will be called a violator space.

Violator operator

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- Consistency of ***V*** is equivalent to ***extensivity*** of φ :
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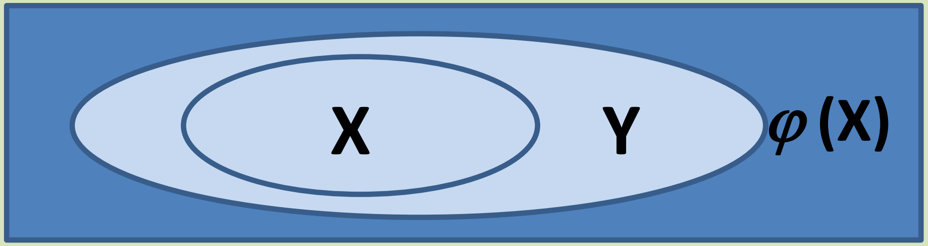
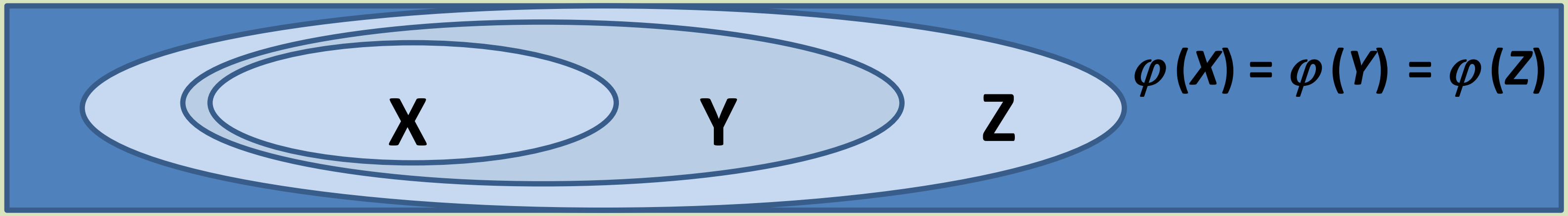


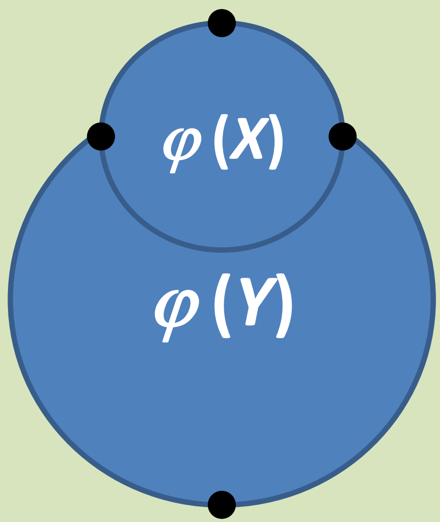
- Violator spaces **(*H*, *φ*)** satisfy **idempotence** and **convexity** :
  $(X \subseteq Y \subseteq Z) \wedge (\varphi(X) = \varphi(Z)) \Rightarrow \varphi(X) = \varphi(Y) = \varphi(Z)$



Violator mappings and closure operators

The axiom of isotonicity not necessarily holds:

Example: $\boldsymbol{H} = \{(x,y) \in \mathbf{Z}^2 \mid -1 \leq x \leq 1, -5 \leq y \leq 5\}$



$\boldsymbol{X}$**={(-1,0),(1,0)}**
$\boldsymbol{Y}$ **={(-1,0),(1,0),(0,-4)}**

$\boldsymbol{X \subseteq Y}$**, but** $\boldsymbol{\varphi(X) \not\subset \varphi(Y)}$
$(0,1) \in \varphi(X)$, $(0,1) \notin \varphi(Y)$

Spanoids- as an abstraction of spanning structures: Definition

Spanoids were introduced by Dvir, Goppy, Gu, Wigderson (2020) as logical inference structures with applications in several areas including coding theory.

Consider the following implication $\boldsymbol{T} \rightarrow \boldsymbol{i}$ with T $\subseteq$ [n] and i $\in$[n]: the values of codewords in coordinate positions T, determine the value of some other coordinate i.

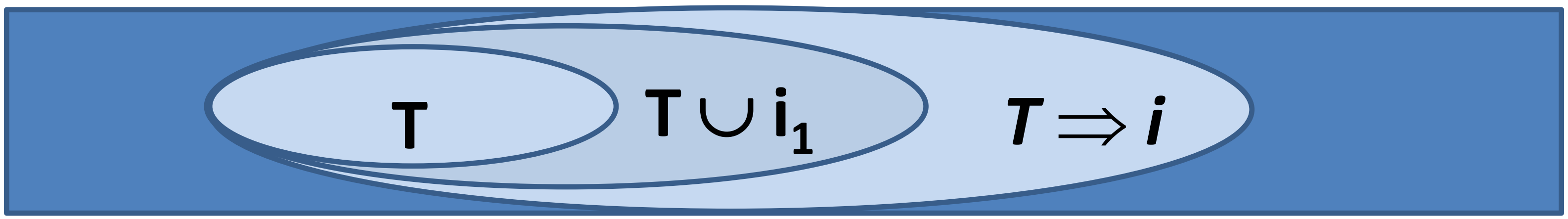
A ***Spanoid*** over [n] is a family of pairs $\boldsymbol{T} \rightarrow \boldsymbol{i}$, $\boldsymbol{T} \subseteq$ [n], $\boldsymbol{i} \in$ [n] s.t:

- $\boldsymbol{i} \rightarrow \boldsymbol{i}$ for each $\boldsymbol{i} \in$ [n]
- for all $\boldsymbol{X} \subseteq \boldsymbol{Y}$: $\boldsymbol{X} \rightarrow \boldsymbol{i} \Rightarrow \boldsymbol{Y} \rightarrow \boldsymbol{i}$ (monotonicity)

Spanoids

A **derivation** $\boldsymbol{T \Rightarrow i}$ is a chain of sets $\boldsymbol{T=T_0,T_1,\ldots,T_r}$, where for each $0<j\leq r$ there is $\boldsymbol{i_j \in E}$ such that $\boldsymbol{T_{j-1} \rightarrow i_j}$,and $\boldsymbol{T_j=T_{j-1} \cup i_j}$, and $\boldsymbol{T_r=T_{r-1} \cup i}$.

$$\mathbf{T=T_0 \underset{i_1}{\rightarrow} T_1=T_0 \cup i_1 \underset{i_2}{\rightarrow} T_2=T_1 \cup i_2 \rightarrow \ldots \rightarrow T_r= T_{r-1} \cup i}$$



span(T)={i | $T \Rightarrow i$}

span is a closure operator

Spanoids: Definition

Example 1: Graphic **Matroid**

- ***E*** is the set of edges of a graph G=(V,E)
- F = { A $\subseteq$ E | T = G[A] is an induced subgraph of G such that **T is a forest** } – independent sets.

- M(G)=(E,F) is a (graphic) matroid:

  M1: $\varnothing \in$ **F**

  M2: If **A**$\in$**F** and **B** $\subseteq$ **A**, then **B**$\in$**F** (hereditary)

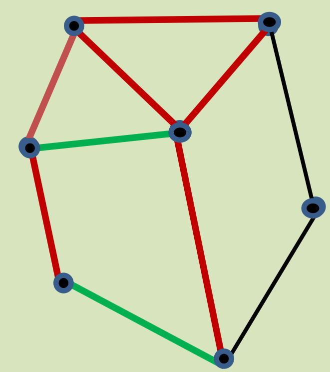
  M3: If **A,B**$\in$**F** and **|A|<|B|** $\Rightarrow \exists$ **x** $\in$ **B\A s. t. A**$\cup$**{x}** $\in$ **F**

  (augmentation property)

Spanoids: Definition

Example 1: Graphic **Matroid**

- ***E*** is the set of edges of a graph $G=(V,E)$
- F = { A $\subseteq$ E | T = G[A] is an induced subgraph of G such that **T is a forest** }.
- M(G)=(E,F) is a graphic matroid.
- **rank(X)** - is the size of the largest forest in X $\Leftrightarrow$ the maximum number of edges which do not close a cycle



- $X \rightarrow i \Leftrightarrow$ **rank(X)=rank($X \cup i$)**
- **span(X)={i|rank(X)=rank($X \cup i$)}**

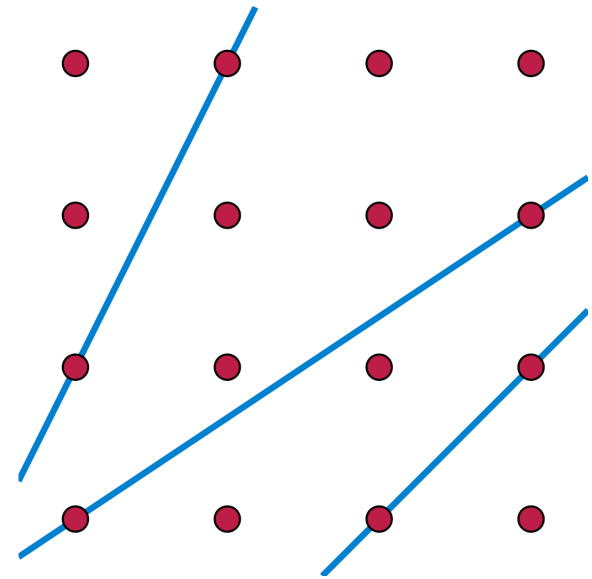
Spanoids: Example

Example 2:

- $\boldsymbol{E}$ is the set of $\boldsymbol{n}$ points in Euclidean space $\boldsymbol{R}^d$
- $\boldsymbol{X} \rightarrow \boldsymbol{q} \iff \boldsymbol{q}$ lies on a line passing through two points from $\boldsymbol{X}$

Sylvester-Gallai theorem:

- If the line through any two points passes through a third point, then they must all be collinear
- If for any two points $\boldsymbol{p,t} \in \boldsymbol{E}$ there is a third point $\boldsymbol{q} \in \boldsymbol{E}$ s.t. $\{\boldsymbol{p,t}\} \rightarrow \boldsymbol{q}$, then all points are collinear
(the span is a one-dimensional space)



Greedoids: Definition

Greedoids were introduced by Korte and Lovasz (1981) as a generalization of matroids.

A ***Greedoid*** over ***E*** is a family of feasible sets $F \subseteq 2^E$ such that:

- $\varnothing \in F$
- if $X, Y \in F$ and $|X|<|Y|$ then there exists a $y \in Y - X$ such that $X \cup y \in F$ (augmentation property)

Greedoids: Definition

Greedoids were introduced by Korte and Lovasz (1981) as a generalization of matroids.

A ***Greedoid*** over $\boldsymbol{E}$ is a family of feasible sets $\boldsymbol{F} \subseteq \mathbf{2^E}$ such that:

- $\varnothing \in \boldsymbol{F}$
- if $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{F}$ and $|\boldsymbol{X}|<|\boldsymbol{Y}|$ then there exists a $\boldsymbol{y} \in \boldsymbol{Y} - \boldsymbol{X}$ such that $\boldsymbol{X} \cup \boldsymbol{y} \in \boldsymbol{F}$ (augmentation property)

$$\mathrm{rank}(X)=\max\{|A|: A \subseteq X, A \in F\}$$

$$\sigma(X)=\{x: \mathrm{rank}(X \cup x)= \mathrm{rank}(X)\}$$

Closure of greedoids

$$\text{rank}(X)=\max\{|A|: A \subseteq X, A \in F\}$$

$$\sigma(X)=\{x: \text{rank}(X \cup x)= \text{rank}(X)\}$$

- for all $X \subseteq E$, $X \subseteq \sigma(X)$ (*extensivity*)
- $X \subseteq Y \subseteq \sigma(X) \rightarrow \sigma(X) = \sigma(Y)$ *self-convexity*
- $x,y \notin X$ **and** $X \cup x \in F \rightarrow x \in \sigma(X \cup y) \Rightarrow y \in \sigma(X \cup x)$ -
  a weaker version of the Steinitz- MacLine exchange property
- **σ is not necessarily isotone**

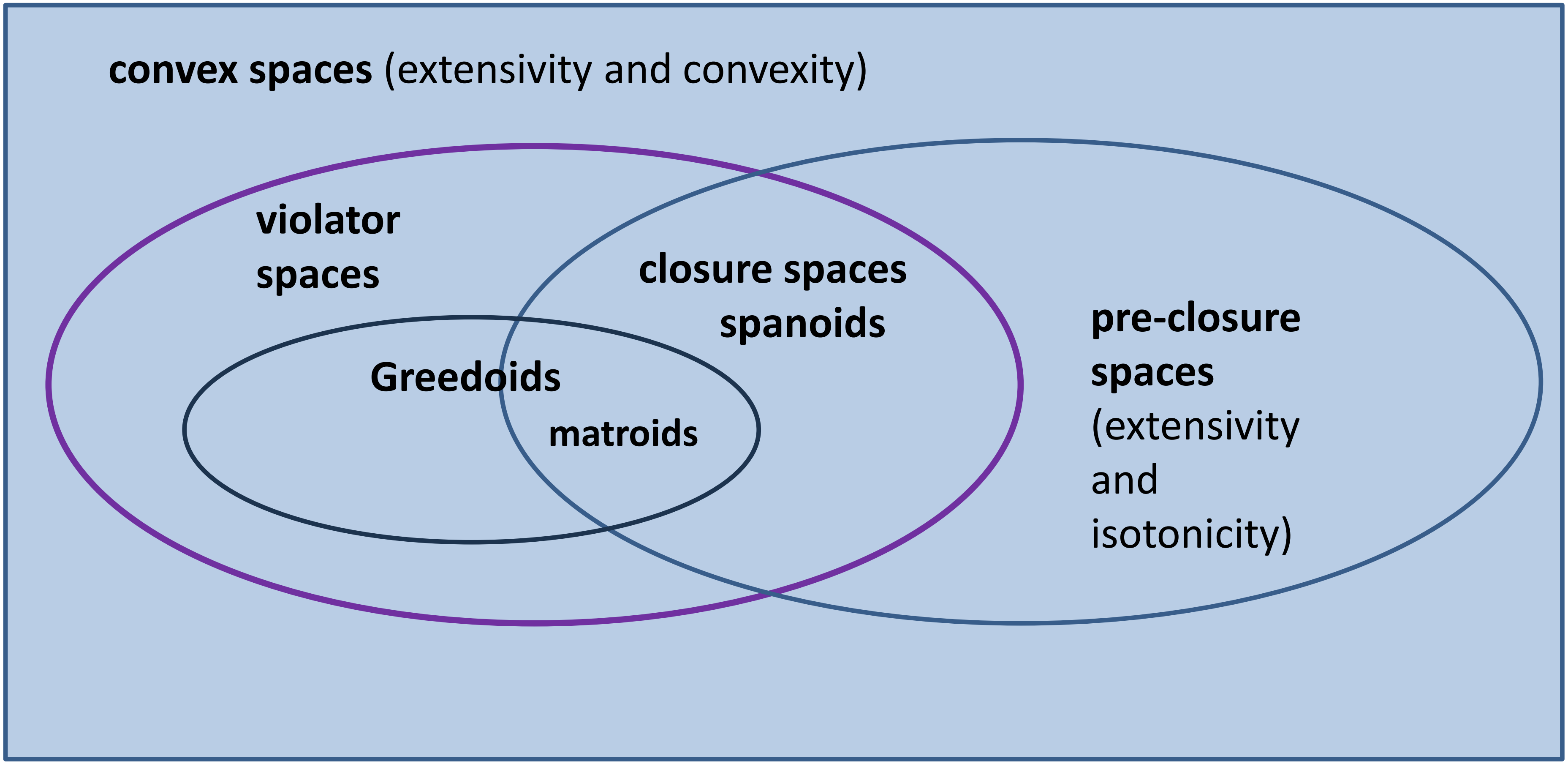
Violator spaces vs Greedoids

Violator spaces $(\boldsymbol{H}, \boldsymbol{\varphi})$ satisfy
- extensivity
- self-convexity
- idempotence
- convexity
- not necessarily isotone

Greedoids $(\boldsymbol{E}, \boldsymbol{\sigma})$ satisfy
- extensivity
- self-convexity
- idempotence
- convexity
- not necessarily isotone
- $x,y \notin X$ **and** $X \cup x \in F \rightarrow$
  $\mathbf{x} \in \sigma (\mathbf{X} \cup \mathbf{y}) \Rightarrow \mathbf{y} \in \sigma (\mathbf{X} \cup \mathbf{x})$

Greedoids may be considered as a subclass of violator spaces

Spanoids, Greedoids and Violator spaces



Thank you!

Hvala vam!

- Dvir, Z., Gopi, S., Gu, Y., & Wigderson, A.(2020)

Spanoids - an abstraction of spanning structures, and a barrier for LCCs.

SIAM Journal on Computing, *49*(3),

465-496.

https://doi.org/10.1137/19M124647X

Algorithmic characterizations

The class of violator spaces is the most general one for which Clarkson's algorithm is still guaranteed to work (Škovrŏn, 2007)

The greedy algorithm maximizes some bottleneck function on a set system if and only if the set system is an antimatroid (dual to convex geometry) (Boyd, 1990; Kempner, Levit, 2003).

Open problems:

- Applications of Clarkson's algorithm to closure systems
- Algorithmic characterizations of closure systems