Balanced designs related to projective planes

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Balancedly Splittable Hadamard matrices

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$$H = \left[\begin{array}{c} H_1 \\ \hline H_2 \end{array}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{array}\right]$$

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$$H_1^t H_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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$$H_2^t H_2 = \begin{bmatrix} 3 & - & - & - \\ - & 3 & - & - \\ - & - & 3 & - \\ - & - & - & 3 \end{bmatrix}$$

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Every normalized Hadamard matrix is balancedly splittable in this way.

Here is a twin balancedly splitted Hadamard matrix of order 16:

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$$\left[\begin{array}{c} H_0 \\ \hline H_1 \\ \hline H_2 \end{array}\right] = \left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
\hline
1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 \\
\hline
1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & - & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - \\
1 & - & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - \\
1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & -
\end{array}\right]$$

$$H_0 = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&-&-&-&-&1&1&1&1&-&-&-&- \\ 1&1&1&1&1&1&1&1&-&-&-&-&-&-&-&- \\ 1&1&1&1&-&-&-&-&-&-&-&-&1&1&1&1 \end{bmatrix}$$

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$$H_1 = \begin{bmatrix} 1&-&1&-&1&-&1&-&1&-&1&-&1&-&1&- \\ 1&1&-&-&1&1&-&-&1&1&-&-&1&1&-&- \\ 1&-&-&1&1&-&-&1&1&-&-&1&1&-&-&1 \\ 1&1&-&-&-&-&1&1&1&1&-&-&-&-&1&1 \\ 1&-&1&-&1&-&1&-&-&1&-&1&-&1&-&1 \\ 1&-&-&1&-&1&1&-&-&1&1&-&1&-&-&1 \end{bmatrix}$$

$$H_0 = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&-&-&-&-&1&1&1&1&-&-&-&- \\ 1&1&1&1&1&1&1&1&-&-&-&-&-&-&-&- \\ 1&1&1&1&-&-&-&-&-&-&-&-&1&1&1&1 \end{bmatrix}$$

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$$H_2 = \begin{bmatrix} 1&-&1&-&-&1&-&1&1&-&1&-&-&1&-&1 \\ 1&-&-&1&-&1&1&-&1&-&-&1&-&1&1&- \\ 1&1&-&-&1&1&-&-&-&-&1&1&-&-&1&1 \\ 1&-&-&1&1&-&-&1&-&1&1&-&-&1&1&- \\ 1&-&1&-&-&1&-&1&-&1&-&1&1&-&1&- \\ 1&1&-&-&-&-&1&1&-&-&1&1&1&1&-&- \end{bmatrix}$$

$$H_0^t H_0 = \begin{bmatrix} 4&4&4&4&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&4&4&4&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&4&4&4&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&4&4&4&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&4&4&4&4&0&0&0&0&0&0&0&0 \\ 0&0&0&0&4&4&4&4&0&0&0&0&0&0&0&0 \\ 0&0&0&0&4&4&4&4&0&0&0&0&0&0&0&0 \\ 0&0&0&0&4&4&4&4&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&4&4&4&4&0&0&0&0 \\ 0&0&0&0&0&0&0&0&4&4&4&4&0&0&0&0 \\ 0&0&0&0&0&0&0&0&4&4&4&4&0&0&0&0 \\ 0&0&0&0&0&0&0&0&4&4&4&4&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4 \\ 0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4 \\ 0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4 \\ 0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4 \end{bmatrix}$$

$$H_1^t H_1 = \begin{bmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 6
\end{bmatrix}
\quad
H_2^t H_2 = \begin{bmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
\bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 6
\end{bmatrix}$$

The corresponding angle between lines is $arccos(\frac{1}{3})$ for both sets of lines.

$$H_0^t H_0 + H_1^t H_1 = \begin{bmatrix}
10 & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & 2 & 2 & 10 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 10 & 2 & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 2 & 10 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 10 & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 10 & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 10 & 2 & 2 & 2 \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 & 2 \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 10 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 2 & 10
\end{bmatrix}$$

The corresponding angle between lines is $arccos(\frac{1}{5})$.

$$H_0^t H_0 + H_2^t H_2 = \begin{bmatrix}
10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & 10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 \\
2 & 2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} \\
\bar{2} & 2 & \bar{2} & 2 & 10 & 2 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 10 & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 \\
\bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 10 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 10 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 \\
\bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 10 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & 2 & 10 & 2 & \bar{2} & 2 & \bar{2} \\
\bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 10 & 2 & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 10 & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 10 & 2 \\
\bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 10
\end{bmatrix}$$

The corresponding angle between lines is $arccos(\frac{1}{5})$.

Definition

A normalized Hadamard matrix H is *balancedly splittable* if by suitably permuting its rows it can be transformed to

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

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$$H_1^t H_1 = \ell I_n + aA + b(J_n - A - I_n).$$

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We will concentrate on the case where $b = -a$.

Equiangular Tight Frames

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Let $X \subset \mathbb{R}^m$ be a set of unit vectors such that $|\langle v, w \rangle| = \alpha$ for all $v, w \in X, v \neq w$. If $m < \frac{1}{\alpha^2}$, then

$$|X| \leq \frac{m(1-\alpha^2)}{1-m\alpha^2}.$$

A balancedly splittable Hadamard matrix

$$H = \left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
\hline
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
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\hline
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
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$$

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The rows of a splitted Hadamard matrix considered as lines in $\mathbb{R}^6$

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$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & - & - \\ - & - & - & - & 1 & 1 \\ 1 & 1 & 1 & - & 1 & - \\ 1 & 1 & - & 1 & - & 1 \\ 1 & - & 1 & 1 & 1 & - \\ - & 1 & 1 & 1 & - & 1 \\ 1 & - & 1 & - & 1 & 1 \\ - & 1 & - & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - & 1 \\ 1 & 1 & - & 1 & 1 & - \\ - & 1 & 1 & 1 & 1 & - \\ 1 & - & 1 & 1 & - & 1 \\ 1 & - & - & 1 & 1 & 1 \\ - & 1 & 1 & - & 1 & 1 \end{pmatrix}$$

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Definition: Unbiased Hadamard Matrices

Hadamard matrices H and K of order n are unbiased if

$$HK^t = \sqrt{n}L$$

for some Hadamard matrix L of order n.

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\end{array}\right)$$

- a balanced split

From the splitted matrix H another matrix K

$$K = \left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
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1&1&1&1&1&1&1&1&-&-&1&1&1&1&-&-\\
1&-&1&-&-&-&-&1&-&1&1&1&1&-&-&1\\
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-&1&-&1&-&-&-&1&1&-&1&1&1&-&1&-\\
-&1&-&1&-&-&1&-&-&1&1&1&-&1&-&1\\
-&-&1&1&1&-&-&-&-&1&1&-&1&1&1&-\\
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- is formed
- H and K are unbiased

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Some properties of balancedly splitted Hadamard matrices

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Let $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ be a balancedly splitted Hadamard matrix of order n with $H_1^t H_1 = \ell I_n + aS$ where $a \neq 0$ and S is an $n \times n$ $(0, 1, -1)$-matrix.

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In this case, $n = 4k^2$ for some integer k,

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Two of the five Hadamard matrices of order 16 fail to be balancedly splittable with $(\ell, a) = (6, 2)$.

Nonexistence

There is no balancedly splittable Hadamard matrix with the parameters (n, ℓ, a), $\ell + a \not\equiv 0 \pmod 4$.

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Let x, y, x, w be non-negative integers such that

$$\begin{aligned}
\text{the first column} &= (\underbrace{+\cdots+}_{x \text{ rows}} \quad \underbrace{+\cdots+}_{y \text{ rows}} \quad \underbrace{+\cdots+}_{z \text{ rows}} \quad \underbrace{+\cdots+}_{w \text{ rows}})^\top, \\
\text{the } i\text{-th column} &= (+\cdots+ \quad +\cdots+ \quad -\cdots- \quad -\cdots-)^\top, \\
\text{the } j\text{-th column} &= (\underbrace{+\cdots+}_{x \text{ rows}} \quad \underbrace{-\cdots-}_{y \text{ rows}} \quad \underbrace{+\cdots+}_{z \text{ rows}} \quad \underbrace{-\cdots-}_{w \text{ rows}})^\top.
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\text{the first column} &= (\underbrace{+\cdots+}_{x \text{ rows}} \quad \underbrace{+\cdots+}_{y \text{ rows}} \quad \underbrace{+\cdots+}_{z \text{ rows}} \quad \underbrace{+\cdots+}_{w \text{ rows}})^\top, \\
\text{the } i\text{-th column} &= (+\cdots+ \quad +\cdots+ \quad -\cdots- \quad -\cdots-)^\top, \\
\text{the } j\text{-th column} &= (+\cdots+ \quad -\cdots- \quad +\cdots+ \quad -\cdots-)^\top.
\end{aligned}$$

Then it follows that

$$\begin{cases}
x + y + z + w &= \ell, \\
x + y - z - w &= a, \\
x - y + z - w &= a, \\
x - y - z + w &= -a.
\end{cases}$$

There is no balancedly splittable Hadamard matrix with the parameters (n, ℓ, a), $\ell + a \not\equiv 0 \pmod 4$.
Let x, y, x, w be non-negative integers such that

$$\begin{aligned}
\text{the first column} &= (\underbrace{+\cdots+}_{x \text{ rows}} \quad \underbrace{+\cdots+}_{y \text{ rows}} \quad \underbrace{+\cdots+}_{z \text{ rows}} \quad \underbrace{+\cdots+}_{w \text{ rows}})^\top, \\
\text{the } i\text{-th column} &= (+\cdots+ \quad +\cdots+ \quad -\cdots- \quad -\cdots-)^\top, \\
\text{the } j\text{-th column} &= (+\cdots+ \quad -\cdots- \quad +\cdots+ \quad -\cdots-)^\top.
\end{aligned}$$

Then it follows that

$$\begin{cases}
x + y + z + w &= \ell, \\
x + y - z - w &= a, \\
x - y + z - w &= a, \\
x - y - z + w &= -a.
\end{cases}$$

Solving these equations yields $(x, y, z, w) = (\frac{\ell+a}{4}, \frac{\ell+a}{4}, \frac{\ell+a}{4}, \frac{\ell-3a}{4})$.
Therefore, $\ell + a \equiv 0 \pmod 4$.

Existence

Theorem (K, Pender, Suda, DCC 2021)

There is a balancedly splittable Hadamard matrix of order $4n^2$ for any n an order of a Hadamard matrix.

There are nine submatrices forming the desired matrix:

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There are nine submatrices forming the desired matrix:

$$\begin{bmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{bmatrix}.$$

- The most important Hadamard matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}$$

- Auxiliary matrices:

$$c_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad c_1 = \begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}$$

The construction

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S1: **Form the block Barker sequence**

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  is a block Barker sequence with block autocorrelation 0

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S2: **Form the block Golay sequence**

- The sequences

  $$(c_0, c_1, c_1) \qquad (c_0, c_1, -c_1)$$

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- bcirc$(c_0 c_1 c_1)$ bcirc$(c_0 c_1 \bar{c}_1)$
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The construction

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A=bcirc($c_0 c_1 c_1$), B=bcirc($c_0 c_1 \bar{c}_1$)

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A=bcirc$(c_0 c_1 c_1)$, B=bcirc$(c_0 c_1 \bar{c}_1)$

Then the matrix

$$\Theta = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \left(\begin{array}{cccccc|cccccc}
1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)$$

And

$$\Theta\Theta^t = (2)\left(\begin{array}{cccccc|cccccc}
6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
\bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 \\
\hline
2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6
\end{array}\right)$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\hline
* & * & * & * & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
* & * & * & * & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
* & * & * & * & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
* & * & * & * & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
* & * & * & * & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
* & * & * & * & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
* & * & * & * & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
* & * & * & * & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
* & * & * & * & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
* & * & * & * & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
* & * & * & * & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
* & * & * & * & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

$$
\left(\begin{array}{c|cccccc|cccccc}
G & & & F & & & & & & -F & & & \\
\hline
 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
E & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
-E & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
\hline
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
\hline
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right)
$$

Summary

For Hadamard matrices

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- There is a balancedly splittable Hadamard matrix of order $64n^2$ for any $4n$ an order of a Hadamard matrix.

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- No Hadamard matrix of order $4n^2$, n odd, is balancedly splittable.

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Balancedly multi-splittable Hadamard matrices

OA(5,4) on $\{1,2,3,4\}$:

$$\begin{bmatrix} 1\,1\,1\,1\,1 \\ 1\,2\,2\,2\,2 \\ 1\,3\,3\,3\,3 \\ 1\,4\,4\,4\,4 \\ 2\,1\,2\,3\,4 \\ 2\,2\,1\,4\,3 \\ 2\,3\,4\,1\,2 \\ 2\,4\,3\,2\,1 \\ 3\,1\,3\,4\,2 \\ 3\,3\,1\,2\,4 \\ 3\,4\,2\,1\,3 \\ 3\,2\,4\,3\,1 \\ 4\,1\,4\,2\,3 \\ 4\,4\,1\,3\,2 \\ 4\,2\,3\,1\,4 \\ 4\,3\,2\,4\,1 \end{bmatrix}$$

A normalized Hadamard matrix H_4 :

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}$$

We have a 16x16 matrix with an added column of 1's obtained from OA(5,4) and the rows of H_4 from which the first column is removed

$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 \\
1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & - \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - \\
1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 \\
1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 \\
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1 & - & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1
\end{array}\right]
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$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 \\
1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & - \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - \\
1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 \\
1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & - \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - \\
1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 \\
1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 \\
1 & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & - \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - \\
1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - \\
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1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 \\
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Conjecture: Hadamard matrices of order 4^n are the only Hadamard matrices which are possibly BMS.

Hadamard matrices related to projective planes

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Let C be an equidistance code of length $q+1$ over the symbol set $\{1, \ldots, q\}$. Then

$$|C| \leq q^2$$

holds. Equality holds if and only if the matrix whose rows consists of the codewords of C is an orthogonal array $OA(q+1, q)$.

Since the code C attains the upper bound in Lemma A is an orthogonal array $\mathrm{OA}(4n+1, 4n)$.

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Theorem (K, Suda EJC 2023)

Let n be the order of a quaternary Hadamard matrix. The existence of a projective plane of order n is equivalent to the existence of a balancedly multi-splittable quaternary Hadamard matrix of order n^2.

Example

Let $n = 10$. Then the existence of an $OA(11, 10)$ on 10 symbols is equivalent to the existence of a BMS quaternary Hadamard matrix of order 100.

Balanced designs related to projective planes

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For the prime power $p \equiv 3 \pmod 4$ there is a regular BMS partial Hadamard of order $p^2 \times p(p+1)$ that can be extended to a Hadamard matrix of order $p^2 + p$.

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$$\begin{bmatrix}
1&1&1&1&1&1&1&1\\
1&2&2&2&2&2&2&2\\
1&3&3&3&3&3&3&3\\
1&4&4&4&4&4&4&4\\
1&5&5&5&5&5&5&5\\
1&6&6&6&6&6&6&6\\
1&7&7&7&7&7&7&7\\
2&1&2&3&4&5&6&7\\
2&2&3&4&5&6&7&1\\
2&3&4&5&6&7&1&2\\
2&4&5&6&7&1&2&3\\
2&5&6&7&1&2&3&4\\
2&6&7&1&2&3&4&5\\
2&7&1&2&3&4&5&6\\
3&1&3&5&7&2&4&6\\
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3&2&4&6&1&3&5&7\\
3&4&6&1&3&5&7&2\\
3&6&1&3&5&7&2&4\\
4&1&4&7&3&6&2&5\\
4&4&7&3&6&2&5&1\\
4&7&3&6&2&5&1&4\\
4&3&6&2&5&1&4&7\\
4&6&2&5&1&4&7&3\\
4&2&5&1&4&7&3&6\\
4&5&1&4&7&3&6&2\\
5&1&5&2&6&3&7&4\\
5&5&2&6&3&7&4&1\\
5&2&6&3&7&4&1&5\\
5&6&3&7&4&1&5&2\\
5&3&7&4&1&5&2&6\\
5&7&4&1&5&2&6&3\\
5&4&1&5&2&6&3&7\\
6&1&6&4&2&7&5&3\\
6&6&4&2&7&5&3&1\\
6&4&2&7&5&3&1&6\\
6&2&7&5&3&1&6&4\\
6&7&5&3&1&6&4&2\\
6&5&3&1&6&4&2&7\\
6&3&1&6&4&2&7&5\\
7&1&7&6&5&4&3&2\\
7&7&6&5&4&3&2&1\\
7&6&5&4&3&2&1&7\\
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7&4&3&2&1&7&6&5\\
7&3&2&1&7&6&5&4\\
7&2&1&7&6&5&4&3
\end{bmatrix}$$

49x56 multi-splittable partial Hadamard:

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56x56 Hadamard matrix

1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−
1	1	1	−	1	−	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−
1	1	1	−	1	−	−	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1
1	1	1	−	1	−	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−
1	1	1	−	1	−	−	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1
1	1	1	−	1	−	−	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1
1	1	1	−	1	−	−	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	1
−	1	1	1	−	1	−	1	1	1	−	1	−	−	−	1	1	1	−	1	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	1
−	1	1	1	−	1	−	−	1	1	1	−	1	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−
−	1	1	1	−	1	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	−	1	1	1	−	1	−
−	1	1	1	−	1	−	1	−	−	1	1	1	−	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	−	1	1	1	−	1	−	−	−	1	1	1	−	1
−	1	1	1	−	1	−	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	−	1	1	1	−	1	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−
−	1	1	1	−	1	−	1	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	−	1	1	1	−	1	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−	−	1	−	−	1	1	1
−	1	1	1	−	1	−	1	1	−	1	−	−	1	1	1	1	−	1	−	−	−	1	1	1	−	1	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−	−	1	−	−	1	1	1	1	−	1	−	−	1	1
−	−	1	1	1	−	1	1	1	1	−	1	−	−	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	1	−	1	−	−	1	−	1	1	1	−	1	−	1	−	−	1	1	1	−	1	−	1	−	−	1	1
−	−	1	1	1	−	1	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	1	−	1	−	−	1	−	1	1	1	−	1	−	1	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	−	1	−	−
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−	−	1	1	1	−	1	1	1	−	1	−	−	1	−	1	1	1	−	1	−	1	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	1	−	1	−	−	−	−	1	1	1	−	1	−	1	−	−	1	1	1
−	−	1	1	1	−	1	−	1	1	1	−	1	−	1	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	1	−	1	−	−	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	1	−	1	−	−	1
−	−	1	1	1	−	1	1	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	1	−	1	−	−	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	1	1	−	1	−	−	−	1	1	1	−	1	−
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1	−	−	1	1	1	−	1	1	1	−	1	−	−	1	−	−	1	1	1	−	1	1	−	1	−	−	1	−	−	1	1	1	−	1	1	−	1	−	−	1	1	−	1	1	1	−	1	−	−	1	−	−	1	1	1
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1	−	−	1	1	1	−	−	−	1	1	1	−	1	1	−	1	−	−	1	1	−	1	1	1	−	1	−	−	1	−	−	1	1	1	1	1	1	−	1	−	−	1	−	−	1	1	1	−	1	1	−	1	−	−	1
1	−	−	1	1	1	−	1	−	1	−	−	1	1	−	1	1	1	−	1	−	−	1	−	−	1	1	1	1	1	1	−	1	−	−	1	−	−	1	1	1	−	1	1	−	1	−	−	1	−	−	1	1	1	−	1
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1	−	−	1	1	1	−	−	1	−	−	1	1	1	1	1	1	−	1	−	−	1	−	−	1	1	1	−	1	1	−	1	−	−	1	−	−	1	1	1	−	1	1	−	1	−	−	1	1	−	1	1	1	−	1	−
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−	1	−	−	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	1	1	1	1	−	1	−	−	1	−	−	1	1	1	−	1	1	−	−	1	1	1	−	1	1	1	−	1	−	−	−	1	1	1	−	1	−
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−	1	−	−	1	1	1	−	−	1	1	1	−	1	1	−	−	1	1	1	−	1	1	1	−	1	−	−	−	1	−	−	1	1	1	−	1	1	1	−	1	−	1	−	1	−	−	1	1	1	1	−	1	−	−	1
−	1	−	−	1	1	1	1	1	−	1	−	−	1	1	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	−	1	−	−	1	1	−	−	1	1	1	−	−	1	1	1	−	1	−	−	−	1	−	−	1	1	1
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−	1	−	−	1	1	1	−	−	1	1	1	−	1	1	−	−	1	1	1	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	1	1	1	−	1	−	−	−	1	1	1	−	1	−	−	1	−	−	1	1	1
1	−	1	−	−	1	1	1	1	1	−	1	−	−	1	−	1	−	−	1	1	1	−	−	1	1	1	−	−	1	1	1	−	1	−	1	1	−	1	−	−	1	−	1	−	−	1	1	1	−	−	1	1	1	−	1
1	−	1	−	−	1	1	1	−	1	−	−	1	1	1	−	−	1	1	1	−	−	1	1	1	−	1	−	1	1	−	1	−	−	1	−	1	−	−	1	1	1	1	1	1	−	1	−	−	−	1	−	−	1	1	1
1	−	1	−	−	1	1	1	−	−	1	1	1	−	−	1	1	1	−	1	−	1	1	−	1	−	−	1	−	1	−	−	1	1	1	1	1	1	−	1	−	−	1	−	1	−	−	1	1	−	−	1	1	1	−	1
1	−	1	−	−	1	1	−	1	1	1	−	1	−	1	1	−	1	−	−	1	−	1	−	−	1	1	1	1	1	1	−	1	−	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−	1	−	1	−	−	1	1
1	−	1	−	−	1	1	1	1	−	1	−	−	1	−	1	−	−	1	1	1	1	1	1	−	1	−	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−	−	1	1	1	−	1	−	1	−	1	−	−	1	1
1	−	1	−	−	1	1	−	1	−	−	1	1	1	1	1	1	−	1	−	−	−	−	1	1	1	−	1	1	−	−	1	1	1	−	−	1	1	1	−	1	−	1	−	1	−	−	1	1	1	1	−	1	−	−	1
1	−	1	−	−	1	1	−	−	1	1	1	−	1	1	−	−	1	1	1	−	1	−	1	−	−	1	1	1	1	1	−	1	−	−	−	1	1	1	−	1	−	1	1	−	1	−	−	1	−	1	−	−	1	1	1
1	1	−	1	−	−	1	1	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	1	1	−	−	1	1	1	−	−	−	1	1	1	−	1	−	1	1	1	−	1	−
1	1	−	1	−	−	1	1	1	−	1	−	−	1	1	−	1	−	−	1	1	1	−	−	1	1	1	−	−	−	1	1	1	−	1	−	1	1	1	−	1	−	1	1	1	−	1	−	−	1	1	1	−	1	−	−
1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	1	1	−	−	1	1	1	−	−	−	1	1	1	−	1	−	1	1	1	−	1	−	1	1	1	−	1	−	−	1	1	−	1	−	−	1
1	1	−	1	−	−	1	−	1	−	−	1	1	1	1	−	−	1	1	1	−	−	−	1	1	1	−	1	−	1	1	1	−	1	−	1	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1
1	1	−	1	−	−	1	1	−	−	1	1	1	−	−	−	1	1	1	−	1	−	1	1	1	−	1	−	1	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	1
1	1	−	1	−	−	1	−	−	1	1	1	−	1	−	1	1	1	−	1	−	1	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	1	1	−	−	1	1	1	−
1	1	−	1	−	−	1	−	1	1	1	−	1	−	1	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	−	1	−	−	1	1	1	1	−	−	1	1	1	−	−	−	1	1	1	−	1
1	1	1	1	1	1	1	−	−	−	−	−	−	−	1	1	1	1	1	1	1	−	−	−	−	−	−	−	1	1	1	1	1	1	1	−	−	−	−	−	−	−	1	1	1	1	1	1	1	−	−	−	−	−	−	−
1	1	1	1	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−	−	−	−	−	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−	−	−	−	−
1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−	−	−	−	−	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−	−	−	−	−	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−
1	1	1	1	1	1	1	−	−	−	−	−	−	−	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−	−	−	−	−	1	1	1	1	1	1	1	−	−	−	−	−	−	−	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−	−	−	−	−	1	1	1	1	1	1	1	−	−	−	−	−	−	−	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−

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