# Rank 3 graphs and the Delsarte and Hoffman bounds

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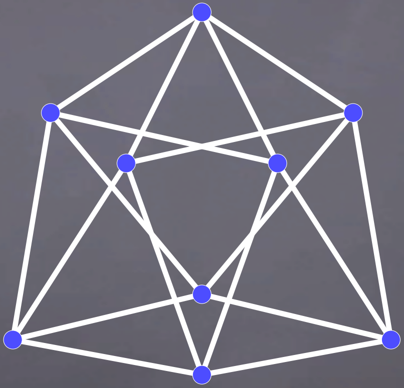
(Joint work with John Bamberg, Michael Giudici and Gordon F. Royle)

# Paley graph $P_9$ | Hamming graph $H(2,3)$ | $3 \times 3$ grid



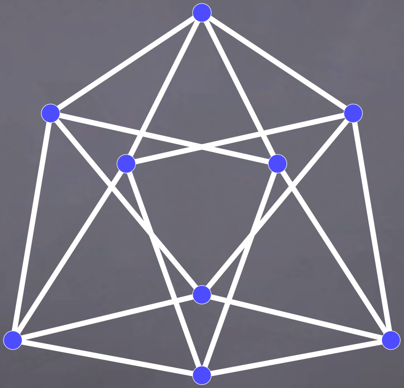
Automorphism group: $S_3 \wr S_2$

## Orbitals of $S_3 \wr S_2$

| | | |
|---|---|---|
| [1,1] | [1,2], [2,1], [1,6], [6,1], | [1,3], [3,1], [1,4], [4,1], |
| [2,2] | [1,7], [7,1], [1,8], [8,1], | [1,5], [5,1], [1,9], [9,1], |
| [3,3] | [2,3], [3,2], [2,7], [7,2], | [2,4], [4,2], [2,5], [5,2], |
| [4,4] | [2,9], [9,2], [3,4], [4,3], | [2,6], [6,2], [2,8], [8,2], |
| [5,5] | [3,8], [8,3], [3,9], [9,3], | [3,5], [5,3], [3,6], [6,3], |
| [6,6] | [4,5], [5,4], [4,7], [7,4], | [3,7], [7,3], [4,6], [6,4], |
| [7,7] | [4,8], [8,4], [5,6], [6,5], | [4,9], [9,4], [5,8], [8,5], |
| [8,8] | [5,7], [7,5], [5,9], [9,5], | [6,7], [7,6], [7,8], [8,7], |
| [9,9] | [6,8], [8,6], [6,9], [9,2], | [7,9], [9,7], [8,9], [9,8]. |
| trivial | edges | non-edges |

Rank 3 graph: Automorphism group has one orbit on edges/non-edges.

# Paley graph $P_9$ | Hamming graph $H(2,3)$ | $3 \times 3$ grid



$$\mathrm{srg}(v, k, \lambda, \mu) = \mathrm{srg}(9, 4, 1, 2)$$

# Eigenvalues of strongly regular graphs

(connected) strongly regular graphs have 3 eigenvalues: $k, r, s$

$$r = \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2},$$

$$s = \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}$$

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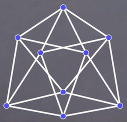

Spectrum: $4^1 1^4 (-2)^4$

# Cliques and the Delsarte bound

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Theorem (Delsarte Bound): $\omega \leqslant 1 - \frac{k}{s}$ for $\mathrm{srg}(v, k, \lambda, \mu)$.

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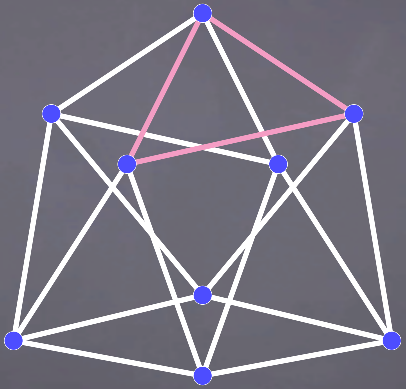
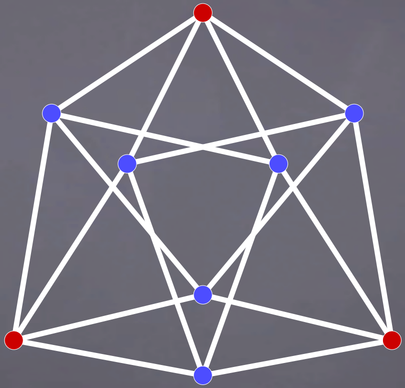


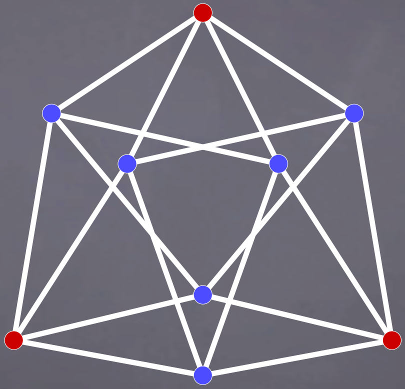
Theorem (Delsarte Bound): $\omega \leqslant 1 - \frac{k}{s}$ for $\mathrm{srg}(v, k, \lambda, \mu)$.

Spectrum: $4^1 1^4 (-2)^4 \to \omega \leqslant 1 - \frac{4}{-2} = 3$.

# Coliques and the Hoffman bound

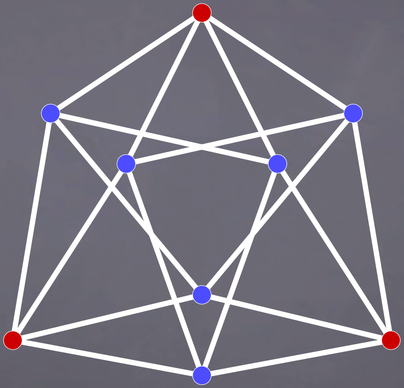


# Coliques and the Hoffman bound



Theorem (Hoffman Bound): $\alpha \leqslant \frac{vs}{s-k}$ for $\mathrm{srg}(v,k,\lambda,\mu)$.

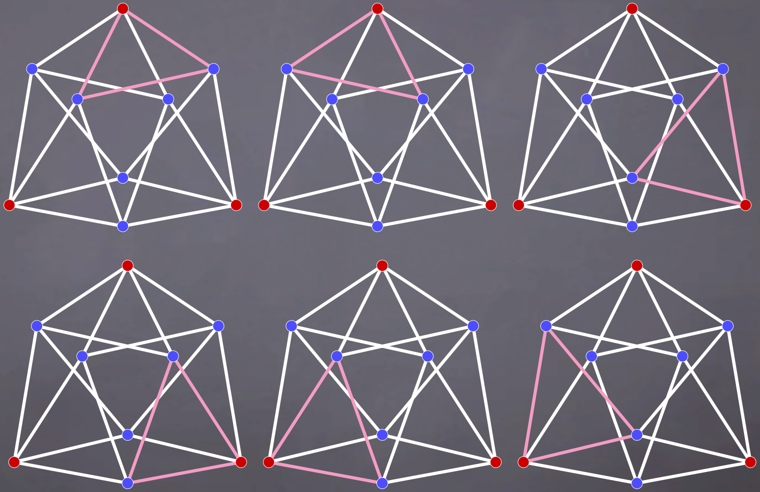
# Coliques and the Hoffman bound



Theorem (Hoffman Bound): $\alpha \leqslant \frac{vs}{s-k}$ for $\mathrm{srg}(v,k,\lambda,\mu)$.

Spectrum: $4^1 1^4 (-2)^4 \to \alpha \leqslant \frac{9(-2)}{-2-4} = 3$.

# Separating graphs



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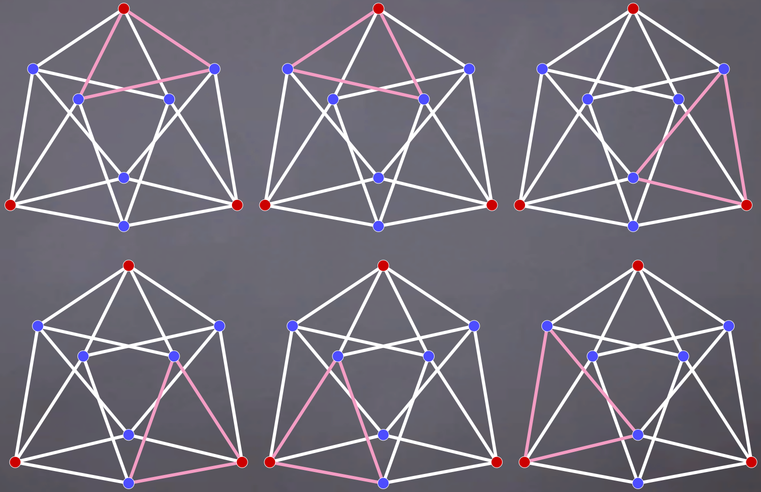


non-separating: Delsarte and Hoffman bounds are both met.

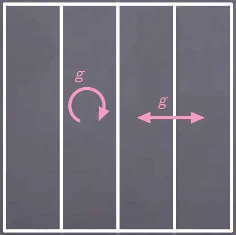
# But why though?

Motivation: synchronisation hierarchy of permutation groups.

$$\begin{array}{c}\text{2-transitive}\\ \Downarrow \\ \text{Separating} \\ \Downarrow \\ \text{Synchronising} \\ \Downarrow \\ \text{Primitive}\end{array}$$

# Separating graphs

(im)primitive



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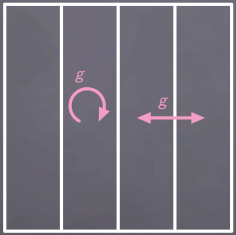
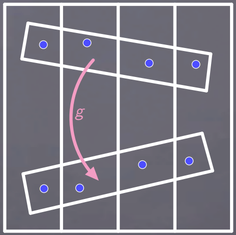
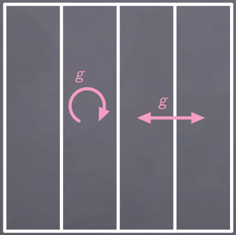



(non)synchronising

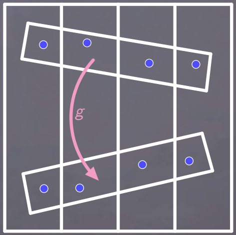


# Separating graphs

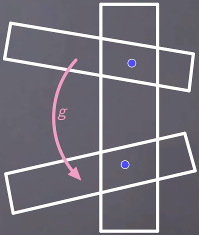
(im)primitive




(non)synchronising




(non)separating



# Synchronisation heirarchy

| | Set version... | |
|---|---|---|
| non-separating | $\exists$ sets $S, T$ s.t. $\lvert S \cap T^g \rvert = 1$, $\forall g \in G$ | |
| $\Uparrow$<br>non-synchronising | $\exists$ partition $\Pi$, set $T$ s.t. $\lvert B \cap T^g \rvert = 1$ $\forall g \in G$, $\forall B \in \Pi$ | |
| $\Uparrow$<br>Imprimitive | $\exists$ partition $\Pi$ s.t. $\Pi^g = \Pi$, $\forall g \in G$ | |

# Synchronisation heirarchy

| | Set version... | Graph version... |
|---|---|---|
| non-separating | $\exists$ sets $S, T$ s.t. $\lvert S \cap T^g \rvert = 1$, $\forall g \in G$ | $\exists$ $G$-invariant $\Gamma$ s.t. $\omega(\Gamma)\alpha(\Gamma) = \lvert V(\Gamma) \rvert$ |
| $\Uparrow$ non-synchronising | $\exists$ partition $\Pi$, set $T$ s.t. $\lvert B \cap T^g \rvert = 1$ $\forall g \in G$, $\forall B \in \Pi$ | $\exists$ $G$-invariant $\Gamma$ s.t. $\chi(\Gamma) = \omega(\Gamma)$ |
| $\Uparrow$ Imprimitive | $\exists$ partition $\Pi$ s.t. $\Pi^g = \Pi$, $\forall g \in G$ | $\exists$ $G$-invariant $\Gamma$ s.t. $\Gamma$ is disconnected |

# Classification of rank 3 primitive groups and graphs

Rank 3 primitive groups are classified (including work by Bannai, Brouwer, Foulser, Kallaher, Kantor, Liebler, Liebeck, Saxl, Soicher, Wilson)

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Corresponding graphs described in Brouwer and Van Maldeghem's new book, *Strongly Regular Graphs*.

> *"aim to give the classification of rank 3 graphs and to describe these graphs ... the project was widened to include the theory of general strongly regular graphs."*
> *- Preface of BVM*

# Classification of rank 3 graphs

**Theorem 2.4** (*cf. [7, Theorems 11.3.1, 11.3.2, 11.3.3, 11.3.4,11.4.1]*). *Let $\Gamma$ be a strongly regular graph admitting a primitive rank 3 group of automorphisms of almost simple or affine type. Then $\Gamma$ is either one of the special cases listed in Table 2, or it belongs to one of the following families:*

(1) *The triangular graph, $T(n)$, for $n \geqslant 4$;* [7, 1.1.7]
(2) *The collinearity graph of a finite classical polar space or the dual of a finite classical polar space, with rank at least 2 or rank exactly 2, respectively;* [7, Thm 2.2.12 & 2.2.19]
(3) *A connected component of the distance-2 graph of the dual polar graph arising from a polar space of rank 5 and order $(q, 1)$;* [7, Thm 2.2.20]
(4) $\mathrm{NU}_m(2)$, *for $m > 3$;* [7, §3.1.6]
(5) $\mathrm{NO}^{\epsilon}_{2m}(q)$, *for $\epsilon = \pm 1$, $m \geqslant 3$, and $q \in \{2, 3\}$;* [7, §3.1.2]
(6) $\mathrm{NO}^{\epsilon}_{2m+1}(q)$, *for $\epsilon = \pm 1$, $m \geqslant 2$ and $q \in \{3, 4, 8\}$;* [7, §3.1.4]
(7) *The Grassmann graph $J_q(n, 2)$ for $n \geqslant 4$;* [7, §3.5.1]
(8) $E_{6,1}(q)$; [7, §4.9]
(9) *The Paley graph, $P_q$;* [7, §1.1.9]
(10) *The Peisert graph, $P^*(p^{2t})$;* [7, §7.3.6]
(11) *The van Lint–Schrijver graph,* $\mathrm{vLS}(p, e, t)$; [7, §7.3.1]
(12) *The $n \times n$ grid;* [7, §1.1.8]
(13) *The Bilinear forms graph $H_q(2, m)$;* [7, §3.4.1]
(14) $\mathrm{VO}^{\epsilon}_{2m}(q)$; [7, §3.3.1]
(15) *The alternating forms graph,* $\mathrm{Alt}(5, p^m)$; [7, §3.4.2]
(16) *The affine half spin graph,* $\mathrm{VD}_{5,5}(q)$; [7, §3.3.3]
(17) $\mathrm{VSz}(q)$, *for $q = 2^{2e-1}$.* [7, §3.3.1]

# (Almost) classification of rank 3 separating graphs (Bamberg, Giudici, JL, Royle, 2023)

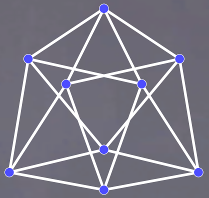
**Theorem 1.1.** *Let $\Gamma$ be a rank 3 graph. Then $\Gamma$ is separating if it or its complement is one of the following:*

(1) *The triangular graph, $T(n)$, for $n \geqslant 5$, $n$ odd;*
(2) *The collinearity graph of a polar space in Table 4;*
(3) *A connected component of the distance-2 graph of the dual polar graph arising from a polar space of rank 5 and order $(q, 1)$;*
(4) $\mathrm{NU}_m(2)$, *for $m > 3$;*
(5) $\mathrm{NO}^{\epsilon}_{2m}(q)$, *for $\epsilon = \pm 1$, $m \geqslant 3$ and $q \in \{2, 3\}$;*
(6) $\mathrm{NO}^{+}_{2m+1}(q)$, *for $m \geqslant 2$, and $q \in \{4, 8\}$;*
(7) $\mathrm{NO}^{-}_{2m+1}(q)$, *for $m \geqslant 2$, $q \in \{3, 4, 8\}$ and $(m, q) \neq (2, 3)$;*
(8) *The Grassmann graph $J_q(n, 2)$, for $n \geqslant 5$, $n$ odd;*
(9) $E_{6,1}(q)$;
(10) $\mathrm{NO}^{+}_{2m+1}(3)$, *for $m \geqslant 3$;*
(11) *The van Lint–Schrijver graph,* $\mathrm{vLS}(p, e, t)$, *for $t$ even;*
(12) $\mathrm{VO}^{-}_{2m}(q)$, *for $m \geqslant 2$;*
(13) $\mathrm{VD}_{5,5}(q)$;
(14) $\mathrm{VSz}(2^{2e+1})$, *for $e \geqslant 0$;*
(15) *$\Gamma$ belongs to Table 1;*

*or is possibly one of the following unresolved cases:*

(I) *The collinearity graph of a polar space not listed in Table 3 or Table 4;*
(II) $\mathrm{VO}^{+}_{2m}(q)$, *for $m > 3$;*
(III) *The Peisert graph $P^*(p^{2t})$, for $t$ even.*

# Paley graphs



**Lemma**

*The Paley graph $P_q$ of order $q$ is separating if an only if $q$ is a non-square.*

**Proof.**

Require $q \equiv 1 \pmod 4$ to be undirected. If $q = q_0^2$ then subfield of order $q_0$ is a clique. Since $P_q$ is self-complementary there is a coclique of the same size. These meet the bounds.

If $q$ is not a square, then $\alpha, \omega < \sqrt{q}$. □

# The hard cases

Non-separation in the collinearity graph of the polar spaces is equivalent to existence of ovoids. Non-separation in $VO^+_{2m}(q)$ is equivalent to existence of ovoids of $Q^+_{2m+1}(q)$.

Peisert graphs are self-complementary: determining one of $\alpha$ or $\omega$ is sufficient.

Conjecture (Yip, 2022): If q is a power of a prime $p \equiv 3 \pmod 4$ and $q > 3$, then the clique number of the Peisert graph of order $q^4$ is strictly less than $q^2$.

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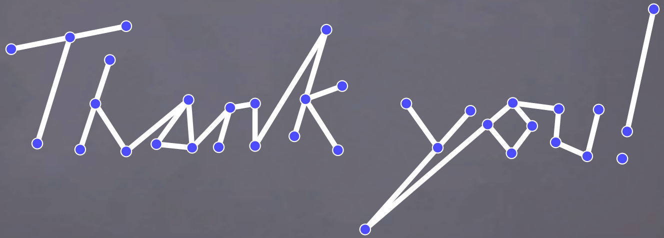
Open problems:

- Resolve existence of ovoids in polar spaces :)
- Determine the clique number of $P^*(q^{2t})$ for $t$ even.


Thank you!