A geometrical picture: semifields and non-singular sublines

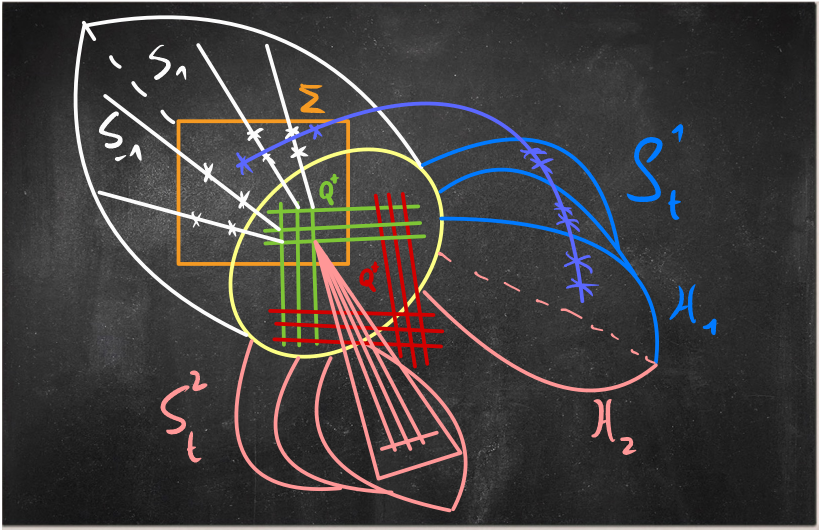
Stefano Lia

RICCOTA, Rijeka

joint work with John Sheekey

UCD, Dublin

A crab-looking picture



Rorschach Test

What do you see in the picture?

- a crab
- an octopus
- a sky rocket
- an hermitian surface, an hyperbolic quadric, a subgeometry in non-canonical position, two families of surfaces giving rise to some quasi-polar spaces.

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Outline

- Finite semifields and their relation with threefold tensors
- A geometric interpretation of the non-singularity of tensors which leads to the picture
- a geometric interpretation of a semifield-invariant
- new quasi-Hermitian surfaces from the picture

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Finite semi-fields

Definition

A finite semi-field $(\mathbb{S}, +, \circ)$ is a finite not-necessarily commutative, not-necessarily associative division algebra.

Theorem

$\mathbb{S}$ a finite semi-field $\implies |\mathbb{S}| = q^n$ a prime power, and it is a vector space of dimension n over a finite field, namely $\mathbb{S} = \mathbb{F}_q^n$.

A common thing is to identify $\mathbb{S}$, namely $\mathbb{F}_q^n$, with $\mathbb{F}_{q^n}$ and define a new product between elements which coincides with the classical if they are in $\mathbb{F}_q$.

Example

Generalized twisted fields (Albert, 1965): $(\mathbb{F}_{q^n}, +, \circ)$ with

$$x \circ y = xy - cx^{q^i} y^{q^j}$$

$N(c) \neq 1$

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Tensors and semifields

Theorem

$$V^\vee \otimes V^\vee \otimes V \simeq \mathrm{Hom}(V \otimes V, V).$$

$\mathrm{Hom}(V \otimes V, V)$ *is precisely the set of* n*-dimensional algebras over* $\mathbb{F}_q$ *(where multiplication is not assumed to be associative). So each tensor defines an algebra, and vice-versa.*

The bilinear form defined by $a^\vee \otimes b^\vee \otimes c$ is the one mapping $x \otimes y$ in $a^\vee(x)b^\vee(y)c$.

Theorem

For every $\mathbb{F}_q$*-bilinear map (multiplication) from* $\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ *to* $\mathbb{F}_{q^n}$ *there exist unique* $c_{i,j} \in \mathbb{F}_{q^n}$ *such that*

$$x \circ y = \sum_{i,j=0}^{n} c_{i,j} x^{q^i} y^{q^j}$$

It turns out that

$$(c_{ij})_{i,j} = \hat{a}^t \hat{b} c$$

where $\hat{a} = (a, a^q, \ldots, a^{q^{n-1}}), \hat{b} = (b, b^q, \ldots, b^{q^{n-1}})$

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General Linear Group actions

In particular, we have that

Yellow Correspondence

$$\textit{Tensors in } (\mathbb{F}_q^n)^{\otimes 3} \iff \textit{matrices } n \times n \textit{ over } \mathbb{F}_{q^n} \iff \textit{points in } PG(n^2-1, \mathbb{F}_{q^n})$$

On tensors of the format $V^\vee \otimes V^\vee \otimes V$, we have a natural action of $GL(n,q) \times GL(n,q) \times GL(n,q)$, given by

$$(a^\vee \otimes b^\vee \otimes c)^{(f,g,h)} := f(a)^\vee \otimes g(b)^\vee \otimes h(c);$$

namely $(x \circ_T y)^{(f,g,h)} = (x^f \circ_T y^g)^h$.

Warning

Under the **Yellow Correspondence**, only f and g will be linear, not h!

The Segre Embedding

The Segre embedding, named after *Corrado* Segre, is the map

$$\sigma = \sigma_{n_1,\ldots,n_l} : \mathrm{PG}(n_1 - 1, \mathbb{K}) \times \cdots \times \mathrm{PG}(n_l - 1, \mathbb{K}) \longmapsto \mathrm{PG}(N - 1, \mathbb{K})$$

where $N = n_1 \cdots n_l$, $\mathbb{K}$ is any field and σ:

$$\sigma(v_1, \ldots, v_l) = v_1 \otimes \cdots \otimes v_l$$

The Segre variety is the image $\mathcal{S}$ of σ.
Rank of a tensor corresponds now to lying in some secant variety of $\mathcal{S}$

Blue Correspondence

Tensors in $(\mathbb{F}_q^n)^{\otimes 3} \iff$ *points in* $PG(n^3 - 1, \mathbb{F}_q)$

Not-Warning

Under the **Blue Correspondence**, everything is linear!

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Contracting tensors

We want to talk about non-singular tensors:

Definition

A vector is non-singular if it is non-zero. Recursively, a tensor is nonsingular if every contraction of it is nonsingular.

A matrix (2-tensor) is nonsingular if whenever you multiply it with a non-zero vector, you get a non-zero vector.

In the case of threefold tensors, this coincides precisely with the associated algebra having no zero divisors (namely: NS-3-tensors $\iff$ semifields!!!)

Observation

You can contract a matrix in 2 ways, but you can contract a threefold tensor in 6 different ways.

The setting of the picture

In our setting:

- $V \simeq \mathbb{F}_q^2 \simeq \mathbb{F}_{q^2}$
- *Tensors in* $(\mathbb{F}_q^2)^{\otimes 3} \iff$ *matrices* 2×2 *over* $\mathbb{F}_{q^2} \iff$ *points in* $PG(3, \mathbb{F}_{q^2})$
- semifields two dimensional over their centre

Nota Bene:

The results on semifields generalize in higher dimension, and are actually interesting mainly there. Here the geometry is nicer and leads to quasi-Hermitian things.

If you ask when a tensor is non-singular, for a particular choice of the contraction, you end up with the system:

$$\begin{cases} Q + Hz + Q^q z^2 \neq 0 \\ z^{q+1} = 1; \end{cases}$$

where $Q = \alpha\delta - \beta\gamma$ and $H = \alpha^{q+1} - \beta^{q+1} - \gamma^{q+1} + \delta^{q+1}$.
Let's call $\mathcal{Q}^+$ and $\mathcal{H}$ the varieties defined in $PG(3, q^2)$ by Q and H respectively.

Very nice fact:

$\mathcal{Q}^+$ and $\mathcal{H}$ are in permutable position!

If $\perp_{\mathcal{Q}^+}$ and $\perp_{\mathcal{H}}$ are the polarities of $\mathcal{Q}^+$ and $\mathcal{H}$, then $\rho = \perp_{\mathcal{Q}^+} \perp_{\mathcal{H}}$ is an involution $\Longrightarrow$ there is a fixed subgeometry Σ

$$\Sigma \cap \mathcal{H} = \Sigma \cap \mathcal{Q}^+ = \mathcal{Q}_0^+$$

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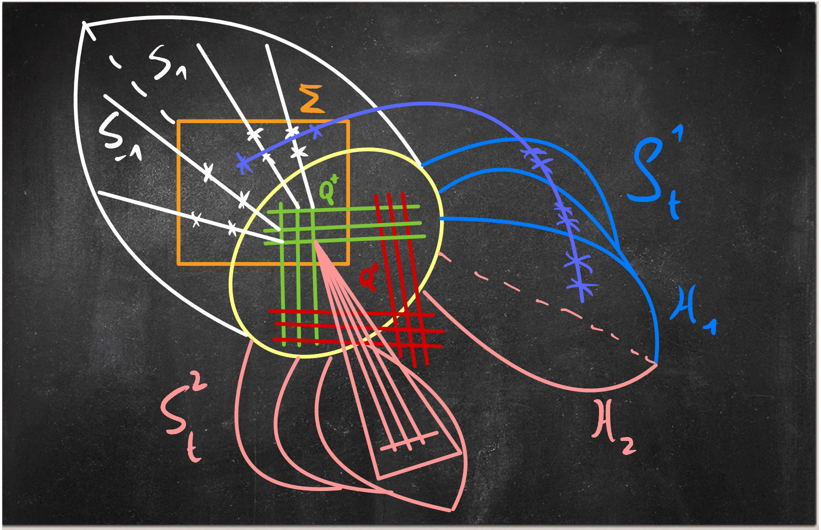

Singularity of tensors

How do I see the contraction of a point in the picture?

- the contraction space is given by

$$c(P) = \{(\alpha z + \delta^q z^q \gamma^q z^q + \beta z\, \gamma z + \beta^q z^q z \alpha^q z^q + \delta z \mid z \in \mathbb{F}_{q^2}\} = \{zP + (zP)^\rho : z \in \mathbb{F}_{q^2}\}$$

- In other words, the contraction space is the unique Σ-subline on which P lies

Lemma (SL, John Sheekey)

P is non-singular if and only if $\Delta(P) \in \square_q^\times$, where $\Delta = H^2 - 4Q^{q+1}$.

- P non-singular $\implies$ each of it's contractions non-singular $\implies$ $c(P)$ does not meet $\mathcal{Q}_0^+$.

Theorem (SL, J. Sheekey)

Non-singular points P are points not in Σ, lying on extended sublines external to $\mathcal{Q}_0^+$. Equivalently, non-singular points are points not in Σ and not on extended $\mathcal{Q}_0^+$-tangent planes.

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Isotopy invariants: BEL-rank

Write

$$x \circ y = \sum_{k=1}^{r} f_k(x) g_k(y)$$

for some $\mathbb{F}_q$-linear maps f_k, g_k, where r is the rank of the matrix $(c_{i,j})$.

Definition

The BEL-rank of a semifield as the minimum such rank across the equivalence(isotopy) class.

Every generalised twisted field has BEL-rank two, as does every semifields two-dimensional over a nucleus.

Theorem (SL-J.Sheekey)

In the picture, semifields are points corresponding to non-singular tensors $\implies$ they are on an extended line $\implies$ they are linear combination of two things of rank one $\implies$ BEL-rank at most 2!

Isotopy invariants: BEL-rank

In the general case, we have:

- A subgeometry $\Sigma = \mathrm{PG}(n^2-1, q)$ of $\mathrm{PG}(n^2-1, q^n)$
- The contraction of a point is again the subspace of Σ of minimum dimension on whose extension the point lies
- All points on one of these extended subspaces (in the same secant variety with respect to the subspace) correspond to equivalent tensors under the full group

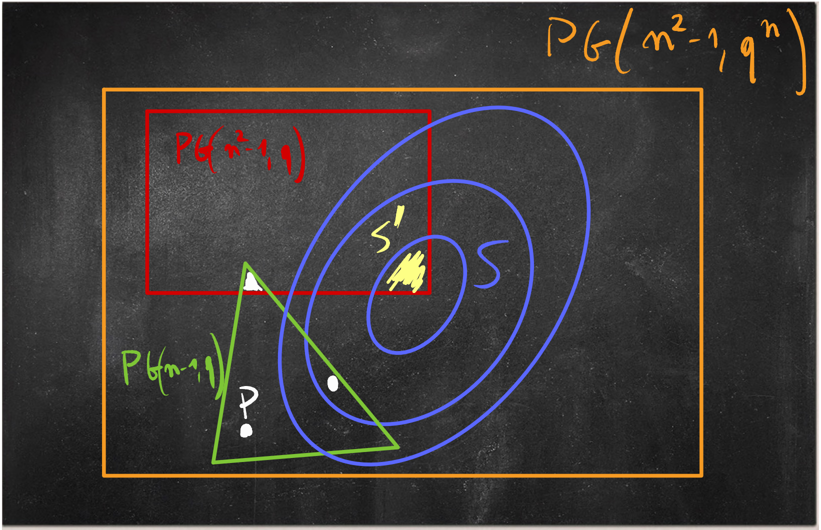
Theorem (SL, J. Sheekey)

The BEL-rank of a finite semifield is at most $n-1$.

Corollary (Dickson, 1905)

Every semifield of dimension two over its centre is isotopic to a field.

The general case



$(Beniamino)$Segre-type problems

Segre 1955

A set of $q+1$ points in $PG(2,q)$, q odd, no three of which are collinear, is a conic.

Moral of the story

A conic, namely a algebraically defined object is characterized by it's combinatorial properties

Which combinatorial properties do we need to characterize a given algebraically defined object?

Example - De Winter - Schillewaert 2010

Let K be a point set of $PG(n,q^2)$, $n>3$, having the same intersection numbers with respect to hyperplanes and codimension two subspaces as the Hermitian variety $H(n,q^2)$. Then K is the point set of $H(n,q^2)$.

What about general polar spaces?

F. De Clerck, N. Hamilton, C. O'Keefe, and T. Penttila 2000

Quadrics are not characterized by their intersections with hyperplanes: there exist quasi-quadrics

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Hermitian varieties are not characterized by their intersections with hyperplanes: there exist quasi-Hermitian varieties

Schillewaert, Van De Voorde 2022

Up to small cases, the size of a polar space is characterized by the intersection with hyperplanes.

Other reasons of interest:

Two character sets $\Longrightarrow$ two-weight codes $\Longrightarrow$ strongly regular graphs.

Therefore

new quasi-polar spaces $\Longrightarrow$ new strongly regular graphs!

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Quasi-Hermitian varieties in $PG(3, q^2)$

1. De Winter, Schillewaert: A note on quasi-Hermitian varieties and singular quasi-quadrics, 2010
2. Aguglia, Cossidente, Korchmaros: On quasi-Hermitian varieties, 2012
3. Aguglia:Quasi-Hermitian varieties in $PG(r, q^2)$, 2013
4. Pavese: Geometric constructions of two-character sets, 2015.
5. Cossidente, Pavese: On line covers of finite projective and polar spaces, 2019
6. Aguglia, Giuzzi: On the equivalence of certain quasi-Hermitian varieties 2022
7. Lavrauw-SL-Pavese: On the geometry of the Hermitian Veronese curve and its quasi-hermitian varieties 2023

Known construction of Quasi-Hermitian surfaces

1. De Winter-Schillewaert 2010–Schillewaert Van De Voorde 2022: pivoting
2. Aguglia, Cossidente, Korchmaros 2012: $\alpha \in \mathbb{F}^*_{q^2}$, $\beta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, with $4\alpha^{q+1} + (\beta^q - \beta)^2 \neq 0$.

$$\mathcal{H}_2 = \{(1, x, y, z) \mid x, y, z \in \mathbb{F}_{q^2}, G(x, y, z) = 0\} \cup \\ \{(0, x, y, x) \mid x, y, z \in \mathbb{F}_{q^2}, x^{q+1} + y^{q+1} = 0\},$$

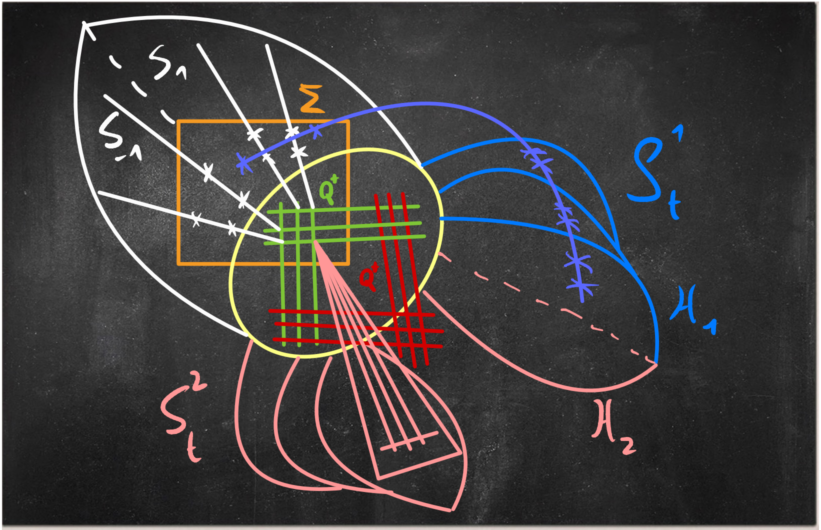
$$G(x, y, z) = z^q - z + \alpha^q(x^{2q} + y^{2q}) - \alpha(x^2 + y^2) - (\beta^q - \beta)(x^{q+1} + y^{q+1})$$

3. Pavese 2015: Σ Baer subgeometry of $\mathrm{PG}(3, q^2)$, $\mathcal{Q}$ a non-degenerate quadric of Σ; $\mathcal{L}$ the set of lines of $\mathrm{PG}(3, q^2)$ having $q + 1$ points on Σ and intersecting $\mathcal{Q}$ in either one or $q + 1$ points.

$$\mathcal{H}_3 = \bigcup_{\ell \in \mathcal{L}} \ell$$

4. Lavrauw, SL, Pavese 2023: joining surfaces in a setting similar to today's setting

The crab is back



Quasi-Hermitian surfaces from the crab

The stabiliser of the small, green subquadric Q_0^+ is isomorphic to $\mathrm{PCGO}^+(4,q)$, and all the objects in the picture are orbits under this group.

In particular, the surfaces S_t^1 and S_t^2 are defined via

$$S_t := \{\langle v\rangle : H - 2tQ^{\frac{q+1}{2}} = 0\}$$

and they partition the points outside of $\mathcal{H} \cup \mathcal{Q}^+ \cup \Sigma$.

The surfaces S_t^1 contain only nonsingular tensors the surfaces S_t^2 contain only singular tensors.

Theorem (SL, J. Sheekey)

For any admissible choice of t_1 and t_2 in $\mathbb{F}_{q^2}$, the join of $S_{t_1}^1$ and $S_{t_2}^2$ is a new quasi-Hermitian surface.

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...and now?

- The picture still requires more explorations: understanding non-singular sublines in our picture implies some understanding of the *four tensors*, whose contractions are NS-sublines
- Is this a happy island or also the general dimension case is so full of nice geometry?
- There are some more geometrical structures related, among which a partition of the subgeometry in quadrics and some relations with the twisted cubic..the picture is much more rich!