Distance-regular graphs with classical parameters which support a uniform structure, $q \leq 1$

(In collaboration with B. Fernández, Š.Miklavič, and G. Monzillo)

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Classify non-bipartite distance-regular graphs with classical parameters with $q \leq 1$ which support a uniform structure.

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We analyze two cases:

- Non-bipartite distance regular graphs of negative type.
- Non-bipartite distance regular graphs with classical parameters with $q = 1$.

Preliminaries

- $\Gamma = (X, \mathcal{R})$: simple, finite, and connected graph,
- $\partial(x, y)$:= distance between x and y, where $x, y \in X$,
- $\varepsilon(x) = max\{\partial(x, y) \,|\, y \in X\}$ (eccentricity of x),
- $D = max\{\varepsilon(x) \,|\, x \in X\}$ (diameter of Γ)
- $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$ (In particular, $\Gamma(x) = \Gamma_1(x)$).
- For an integer $k \geq 0$, we say that Γ is *regular* with valency k whenever $|\Gamma(x)| = k$ for all $x \in X$.

Preliminaries

- Adjacency matrix of Γ defined by

$$(A)_{xy} = \begin{cases} 1 & \partial(x,y) = 1, \\ 0 & \partial(x,y) \neq 1 \end{cases}$$

- V: vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X.
  $M_{|X|}(\mathbb{C})$ acts on V by left multiplication.
  V: the *standard module*
  We endow V with the Hermitian inner product $\langle \, , \, \rangle$ that satisfies $\langle u, v \rangle = u^\top \overline{v}$ for $u, v \in V$.

Preliminaries

Definition 2.1

Fix $x \in X$ *and let* $\varepsilon = \varepsilon(x)$. *For* $0 \leq i \leq \varepsilon$ *let* $E_i^* = E_i^*(x)$ *denote the diagonal matrix in* $M_{|X|}(\mathbb{C})$ *defined by*

$$(E_i^*)_{yy} = \begin{cases} 1 & \partial(x,y) = i, \\ 0 & \partial(x,y) \neq i \end{cases} \qquad (y \in X)$$

E_i^* *is called the* *i-th dual idempotent* *of* Γ *with respect to* x.

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Definition 2.2

Terwilliger algebra $T := T(x)$ *of* Γ, *with respect to* x, *is a subalgebra of* $M_{|X|}(\mathbb{C})$, *generated by the adjacency matrix of* Γ *and the dual idempotents.*

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- $T-$module is a vector subspace W of V, which is invariant for every $t \in T$:

$$tW \subseteq W \text{ for all } t \in T.$$

Preliminaries

Definition 2.3

Let W denote an irreducible T-module. Then, W is an orthogonal direct sum of the nonvanishing spaces among $E_0^*W, E_1^*W, \ldots, E_{D(X)}^*W$. We define

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- *endpoint of W* $\quad r = min\{i|0 \leq i \leq \varepsilon, E_i^*W \neq 0\}$
- *diameter of W* $\quad d = |\{i|0 \leq i \leq \varepsilon, E_i^*W \neq 0\}| - 1$

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In addition,

- Irreducible T-module W is called thin, whenever

$$\dim E_i^*W \leq 1 \qquad \text{for each } 0 \leq i \leq \varepsilon.$$

Preliminaries

Matrices L, F, R

Define $L = L(x), F = F(x)$, and $R = R(x)$ in $M_{|X|}(\mathbb{C})$ by

$$L = \sum_{i=1}^{\varepsilon} E^*_{i-1} A E^*_i, \qquad F = \sum_{i=0}^{\varepsilon} E^*_i A E^*_i, \qquad R = \sum_{i=0}^{\varepsilon-1} E^*_{i+1} A E^*_i.$$

We refer to L, F, and R as the *lowering*, *flat*, and *raising* matrices with respect to x, respectively.

- Note that $L, F, R \in T$, $F = F^\top$, $R = L^\top$, and $A = L + F + R$.

Definition 2.4

A parameter matrix $U = (e_{ij})_{1\le i,j\le \varepsilon}$ *is defined to be a tridiagonal matrix with entries in* $\mathbb{C}$*, satisfying the following properties:*

- $e_{ii} = 1$ $(1 \le i \le \varepsilon)$,
- $e_{i,i-1} \neq 0$ *for* $2 \le i \le \varepsilon$ *or* $e_{i-1,i} \neq 0$ *for* $2 \le i \le \varepsilon$*, and*
- *the principal submatrix* $(e_{ij})_{s\le i,\, j\le t}$ *is nonsingular for* $1 \le s \le t \le \varepsilon$.

For convenience we write $e_i^- := e_{i,i-1}$ *for* $2 \le i \le \varepsilon$ *and* $e_i^+ := e_{i,i+1}$ *for* $1 \le i \le \varepsilon - 1$*.*
We also define $e_1^- := 0$ *and* $e_\varepsilon^+ := 0$*.*

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- Let Γ be a bipartite graph. A uniform structure of Γ with respect to x is a pair (U, f) where $f = \{f_i\}_{i=1}^{\varepsilon}$ is a vector in $\mathbb{C}^\varepsilon$, such that

$$e_i^- RL^2 + LRL + e_i^+ L^2 R = f_i L$$

is satisfied on $E_i^* V$ for $1 \le i \le \varepsilon$

Preliminaries

Definition 2.5

Consider $\Gamma = (X, \mathcal{R})$*: a non-bipartite graph,*
fix $x \in X$ *and let* $\mathcal{R}_f = \mathcal{R} \setminus \{yz \mid \partial(x,y) = \partial(x,z)\}$*.*
We define $\Gamma_f = \Gamma_f(x)$ *to be the graph with vertex set* X *and edge set* $\mathcal{R}_f$*, and we observe that* Γ_f *is bipartite and connected.*

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- The graph Γ supports a uniform structure with respect to x, if Γ_f admits a uniform structure with respect to x.

Preliminaries

We know

- The Terwilliger algebra of the graph Γ and its modules,
- Uniform structure for bipartite graphs,
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☞ **Distance regular graphs with classical parameters**.

PRELIMINARIES

Definition 2.6

- *The graph* Γ *is* *distance-regular* *whenever, for all integers* $0 \leq h, i, j \leq D$ *and all* $x, y \in X$ *with* $\partial(x, y) = h$*, the number* $p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$ *is independent of the choice of* x, y*. The constants* (p^h_{ij}) *are known as the intersection numbers of* Γ*. For convenience,*
  $c_i := p^i_{1\,i-1}$ $(1 \leq i \leq D)$,
  $a_i := p^i_{1i}$ $(0 \leq i \leq D)$,
  $b_i := p^i_{1\,i+1}$ $(0 \leq i \leq D-1)$,
  $k_i := p^0_{ii}$ $(0 \leq i \leq D)$.

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Definition 2.7

A distance regular graph Γ *is called a near polygon whenever* $a_i = a_1 c_i$ *for* $1 \leq i \leq D-1$ *and* Γ *does not contain the complete multipartite graph* $K_{1,1,2}$ *as an induced subgraph.*

Definition 2.8 (Distance-regular graphs with classical parameters)

The graph Γ *is said to have classical parameters* (D, q, α, β) *whenever the intersection numbers of* Γ *satisfy*

$$\begin{cases} c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) & (1 \leq i \leq D), \\ b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) & (0 \leq i \leq D-1) \end{cases}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + q + q^2 + \cdots + q^{j-1}.$$

In this case q *is an integer and* $q \notin \{0, -1\}$.

Preliminaries

The following notation has been used throughout all our results.

Notation.

- Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular non-bipartite graph with diameter $D \geq 3$, intersection numbers b_i $(0 \leq i \leq D-1)$, c_i $(1 \leq i \leq D)$, and eigenvalues $\theta_0 > \theta_1 > \ldots > \theta_D$.
- Fix $x \in X$, and let $T = T(x)$ be the Terwilliger algebra of Γ and E_i^* $(0 \leq i \leq D)$ be the dual idempotents of Γ with respect to x.
- Let L, F, and R denote the corresponding lowering, flat, and raising matrices, respectively.
- Let $T_f = T_f(x)$ be the Terwilliger algebra of Γ_f. Note that T_f is generated by the matrices L, R, and E_i^* $(0 \leq i \leq D)$.

Preliminaries

Some Important Tools: The following results are useful tools in our proofs.

Theorem 1 (P. Terwilliger- 1990)

Let $\Gamma = (X, \mathcal{R})$ be a bipartite graph and fix $x \in X$. Let $T = T(x)$ denote the corresponding Terwilliger algebra. Assume that Γ admits a uniform structure with respect to x. Then, the following assertions hold:

(i) *Every irreducible T-module is thin.*

(ii) *The isomorphism class of any irreducible T-module W is determined by its endpoint and its diameter.*

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Some Important Tools: The following results are useful tools in our proofs.

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(i) *Every irreducible T-module is thin.*

(ii) *The isomorphism class of any irreducible T-module W is determined by its endpoint and its diameter.*

Lemma 1

Consider the above notation. The following statements hold.

(i) *Any T-module W is also a T_f-module.*

(ii) *Any thin irreducible T-module W is also a thin irreducible T_f-module.*

Preliminaries

Definition 2.9

Let $\Gamma = (X, \mathcal{R})$ *denote a distance-regular non-bipartite graph with diameter* $D \geq 3$, *intersection numbers* $b_i\,(0 \leq i \leq D-1)$, $c_i\,(1 \leq i \leq D)$, *and eigenvalues* $\theta_0 > \theta_1 > \ldots > \theta_D$. *The graph* Γ *is tight whenever the equality holds in*

$$\left(\theta_1 + \frac{b_0}{a_1+1}\right)\left(\theta_D + \frac{b_0}{a_1+1}\right) \geq -\frac{b_0 a_1 b_1}{(a_1+1)^2}.$$

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DRGs with classical parameters of negative type

DRGs with classical parameters of negative type that support a uniform structure

DRGs with classical parameters of Negative type that support a uniform structure

$q \leq -2$

Let Γ be a distance-regular graph with classical parameters of negative type.

Question. Which graphs, Γ, in this category support a uniform structure?

We split the analysis of this question into three cases:

DRGs with classical parameters of Negative type that support a uniform structure

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- Case 2. Γ has intersection number $a_1 = 0$.

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- Case 1. Γ has intersection number $a_1 \neq 0$ and is not a near polygon.
- Case 2. Γ has intersection number $a_1 = 0$.
- Case 3. Γ is a near polygon.

CASE 1. Γ HAS INTERSECTION NUMBER $a_1 \neq 0$ AND IS NOT A NEAR POLYGON.

Proposition 1 (Š.Miklavič - 2009)

With reference to Notation, assume that Γ is of negative type with $a_1 \neq 0$ and it is not a near polygon. Then, the following statements hold.

- *Up to isomorphism there is a unique irreducible module with endpoint 1 which is non-thin.*
- *Let W denote a non-thin irreducible T-module with endpoint 1. Pick a non-zero $w \in E_1^* W$. Then, the following vectors form a basis for W:*

$$E_i^* A_{i-1} w \quad (1 \leq i \leq D), \qquad E_i^* A_{i+1} w \quad (2 \leq i \leq D-1). \tag{1}$$

Case 1.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]

With reference to Notation, assume that Γ is of negative type with $a_1 \neq 0$ and it is not a near polygon. Then, Γ does not support a uniform structure with respect to x.

Case 1.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]

With reference to Notation, assume that Γ is of negative type with $a_1 \neq 0$ and it is not a near polygon. Then, Γ does not support a uniform structure with respect to x.

Proof.

- Let W denote a non-thin irreducible T-module with endpoint 1 (which is unique),
- pick a non-zero $w \in E_1^* W$ (W is also a T_f-module),
- let $W' \subseteq W$ be an irreducible T_f-module which contains w,
- using the action of L and R on the basis form Proposition 1, we showed that the vectors Rw and LR^2w are linearly independent.
- W' is non-thin,
- by Theorem 1, Γ does not support a uniform structure.

Case 2. Γ has intersection number $a_1 = 0$.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
With reference to Notation, assume that Γ is of negative type with $a_1 = 0$. Then, Γ does not support a uniform structure with respect to x.

Case 3. Γ is a near polygon.

We first recall following results for distance-regular graphs of negative type with $a_1 \neq 0$ and $c_2 > 1$.

Theorem 2 (Chih-wen Weng - 1999)

With reference to Notation, assume Γ has classical parameters (D, q, α, β) where $D \geq 4$. Suppose $q \leq -2$, $a_1 \neq 0$, and $c_2 > 1$. Then, one of the following hold.

- *Γ is the dual polar graph ${}^2A_{2D-1}(-q)$.*
- *Γ is Hermitian forms graph $Her_{-q}(D)$.*
- *$\alpha = (q-1)/2$, $\beta = -(1+q^D)/2$, and $-q$ is a power of an odd prime.*

Corollary 1

With reference to Notation, assume Γ has classical parameters (D, q, α, β). Suppose Γ is a regular near polygon with $q \leq -2$. Then, either Γ is the dual polar graph ${}^2A_{2D-1}(-q)$ or $D = 3$.

Case 3. Γ is a near polygon.

Therefore, we have the following result.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
With reference to Notation, let Γ denote the dual polar graph ${}^2A_{2D-1}(-q)$. Then,

$$-\frac{q^4}{q^2+1}RL^2 + LRL - \frac{q^{-2}}{q^2+1}L^2R = (-q)^{2D-1}L \quad \text{(C. Worawannotai - 2013)}$$

is satisfied on E_i^*V for $1 \leq i \leq D$. Therefore, Γ supports a uniform structure with respect to x, where $e_i^- = -q^4/(q^2+1)$ $(2 \leq i \leq D)$, $e_i^+ = -q^{-2}/(q^2+1)$ $(1 \leq i \leq D-1)$, and $f_i = (-q)^{2D-1}$ $(1 \leq i \leq D)$.

DRGs with classical parameters with $q = 1$

Distance-regular graphs with classical parameters with $q = 1$

Distance-regular graphs with classical parameters with $q = 1$

We have the following classification for DRGs with classical parameters with $q = 1$.

Theorem 3 (Theorem 6.1.1 - Brouwer, Cohen, and Neumaier)

Let Γ denote a distance-regular graph with classical parameters with $q = 1$. Then, Γ is one of the following graphs:

- *Johnson graph $J(n, D)$, $n \geq 2D$, (tight: $n = 2D$)*
- *Gosset graph, (tight)*
- *Hamming graph $H(D, n)$,*
- *Halved cube $\frac{1}{2}H(n, 2)$, (tight: n even)*
- *Doob graph $D(n, m)$, $n \geq 1$, $m \geq 0$.*

We analyze each of these families in order to see which one admits a uniform structure.

Distance-regular graphs with classical parameters with $q = 1$

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
With reference to Notation, let Γ denote a tight graph with classical parameters with $q = 1$. Then, Γ does not support a uniform structure with respect to x.

Corollary 2

If Γ is one of the following graphs,

1. *Johnson graph $J(2D, D)$,*
2. *Gosset graph,*
3. *Halved cube $\frac{1}{2}H(n, 2)$ with n even,*

then, Γ does not support a uniform structure with respect to x.

Johnson graphs $J(n, D)$ with $n > 2D$

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
With reference to Notation, let $\Gamma = J(n, D)$ with $n \geq 2D$. Then, Γ does not support a uniform structure.

Hamming graph $H(D, n)$ with $n \geq 3$

Theorem [B. Fernández, R.Maleki, Š.Miklavič, G. Monzillo - 2023]
With reference to Notation, let Γ denote the Hamming graph $H(D, n)$ with $n \geq 3$. Then,

$$-\frac{1}{2}RL^2 + LRL - \frac{1}{2}L^2R = (n-1)L$$

is satisfied on E_i^*V for $1 \leq i \leq D$ and Γ supports a uniform structure with respect to x, where $e_i^- = -\frac{1}{2}$ $(2 \leq i \leq D)$, $e_i^+ = -\frac{1}{2}$ $(1 \leq i \leq D-1)$, and $f_i = n-1$ $(1 \leq i \leq D)$.

Halved cubes $\frac{1}{2}H(n,2)$ with n odd.

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]
With reference to Notation, let Γ denote the Halved cube $\frac{1}{2}H(n,2)$ with n odd, $n \geq 7$. Recall that $D = \lfloor \frac{n}{2} \rfloor = (n-1)/2$. Then,

$$e_i^- RL^2 + LRL + e_i^+ L^2 R = f_i L$$

is satisfied on $E_i^* V$ for $1 \leq i \leq D$, where

$$e_i^- = \frac{4i-1-2D}{6-8i+4D} \ (2 \leq i \leq D) \qquad e_i^+ = \frac{4i-5-2D}{6-8i+4D} \ (1 \leq i \leq D-1)$$

$$f_i = -(4i-5)(4i-1) + (16i-12)D - 4D^2 \ \ (1 \leq i \leq D).$$

Therefore, Γ supports a uniform structure with respect to x.

Doob graphs $D(n, m)$ where $n \geq 1$, $m \geq 0$

Theorem [B. Fernández, R.M., Š.Miklavič, G. Monzillo - 2023]

With reference to Notation, let Γ denote the Doob graph $D(n, m)$ with $n \geq 1$, $m \geq 0$ and $D = 2n + m \geq 3$. Then,

$$-\frac{1}{2}RL^2 + LRL - \frac{1}{2}L^2R = 3L$$

is satisfied on E_i^*V for $1 \leq i \leq D$ and Γ supports a strongly uniform structure with respect to x, where $e_i^- = -\frac{1}{2}\,(2 \leq i \leq D)$, $e_i^+ = -\frac{1}{2}\,(1 \leq i \leq D-1)$, and $f_i = 3\,(1 \leq i \leq D)$.

Summary of the results

Non-bipartite distance-regular graphs with classical parameters (D, q, α, β) with $q \leq 1$.

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$$\begin{cases} \Gamma \text{ has intersection number } a_1 \neq 0 \text{ and is not a near polygon } \times \\ \Gamma \text{ has intersection number } a_1 = 0 \ \times \\ \\ \Gamma \text{ is a near polygon} \\ \end{cases}$$

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Non-bipartite distance-regular graphs with classical parameters (D, q, α, β) with $q \leq 1$.

- $q \leq -2$ (negative type)

$$\begin{cases} \Gamma \text{ has intersection number } a_1 \neq 0 \text{ and is not a near polygon } \times \\ \Gamma \text{ has intersection number } a_1 = 0 \ \times \\ \Gamma \text{ is a near polygon } \begin{cases} \Gamma \text{ is a dual polar graph, } D \geq 4 \ \checkmark \\ \Gamma \text{ has diameter } D = 3 \ ? \end{cases} \end{cases}.$$

- $q = 1$

$$\begin{cases} \text{Johnson graph } J(n, D),\ n \geq 2D \ \times \\ \text{Gosset graph } \times \\ \text{Hamming graph } H(D, n) \text{ where } n \geq 3 \ \checkmark \\ \text{Halved cube } \frac{1}{2}H(n, 2) \\ \\ \end{cases}$$

Summary of the results

Non-bipartite distance-regular graphs with classical parameters (D, q, α, β) with $q \leq 1$.

- $q \leq -2$ (negative type)

$$\begin{cases} \Gamma \text{ has intersection number } a_1 \neq 0 \text{ and is not a near polygon } \times \\ \Gamma \text{ has intersection number } a_1 = 0 \ \times \\ \Gamma \text{ is a near polygon } \begin{cases} \Gamma \text{ is a dual polar graph, } D \geq 4 \ \checkmark \\ \Gamma \text{ has diameter } D = 3 \ ? \end{cases} \end{cases}.$$

- $q = 1$ $$\begin{cases} \text{Johnson graph } J(n, D),\ n \geq 2D \ \times \\ \text{Gosset graph } \times \\ \text{Hamming graph } H(D, n) \text{ where } n \geq 3 \ \checkmark \\ \text{Halved cube } \frac{1}{2}H(n, 2) \begin{cases} n \text{ even } \times \\ n \geq 7 \text{ odd } \checkmark \end{cases} \\ \text{Doob graph } D(n, m) \text{ with } n \geq 1,\ m \geq 0 \ \checkmark \end{cases}$$


Thank you!

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Remark 1

Examples of regular near polygons of negative type with $D = 3$:

- *Triality graph* $^3D_{4,2}(-q)$
- *Witt graph* M_{24}
- *extended ternary Golay code graph.*

! We don't know if this is a complete list of graphs.

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Theorem 4 (Go and Terwilliger -2002)

Let Γ *denote a distance-regular graph, then the following statements are equivalent:*

(i) Γ *is tight,*

(ii) *every irreducible* T*-module with endpoint* 1 *is thin with local eigenvalue* $\tilde{\theta}_1$ *or* $\tilde{\theta}_D$*,*

(iii) $a_D = 0$ *and every irreducible* T*-module with endpoint* 1 *is thin.*