Refined Enumeration of the Catalan Family of Alternating Sign Matrices

Ivica Martinjak

Zagreb, Croatia

(joint work with Ana Mimica)

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Definition

(Alternating sign matrix) An alternating sign matrix (ASM) is a matrix of -1s, 0s and 1s for which the sum of entries in each row and in each column is equal to 1 and the non-zero entries of each row and of each column alternate in sign.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & - & 1 & 0 & 0 & 0 \\ 1 & - & 1 & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & - & 1 & - & 1 & 0 \\ 0 & 0 & 1 & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

A method of evaluating determinant called condensation reveals ASMs.

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

In particular,

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \to \begin{pmatrix} ae - bc & bf - ce \\ dh - ge & ei - fh \end{pmatrix} \to$$

$$(1)aei+(-1)afh+(-1)bdi+(0)bde^{-1}fh+(1)bfg+(1)cdh+(-1)ceg$$

We interprete terms as -1, 0 and 1 to get ASMs, eg.

$$(1)afh \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The ASM Conjecture

$A_3 = 7$

1, 2, 7, 42, 429, 7463, 218348, 10850216, ...

- A_n graws fast
- having small factors, eg. $10850216 = 2^3 \cdot 13 \cdot 17^2 \cdot 19^2$

Mills, Robbins, Ramsey: The ASM Conjecture,

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Refined ASM Conjecture

Statistics on 1s in the first row

1				
1	1			
2	3	2		
7	14	14	7	
42	105	135	105	42

$$\vdots$$

$$A_{n,i} = \binom{n+i-2}{n-1} \frac{(2n-i-1)!}{(n-i)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.$$

ASM Conjecture Solved

- D. Zeilberger, 1996. proof based on the partition theory and symmetric functions
- G. Kuperberg, 1996. proof based on the Young-Baxter equation
  6-vertex lattice model in statistical mehanics
- D. Zeilberger, proof. based on q-calculus and WZ method

Refined ASM conjecture solved by D. Zeilberger 1996.

ASM and other structures

- square ice model
- Aztec diamonds
- plane partitions

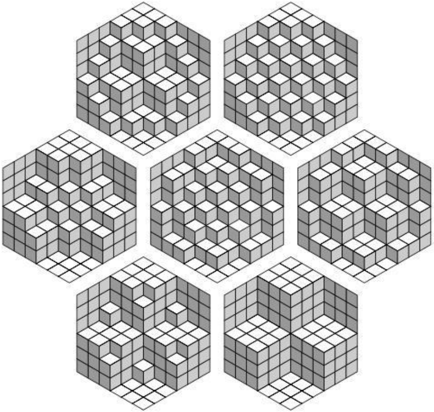
Descending plane partitions (DPP), eg.

$$\begin{array}{cccccc} 7 & 7 & 6 & 5 & 3 & 1 \\ & 6 & 5 & 4 & 2 & \\ & & 3 & 3 & & \\ & & & 2 & & \end{array}$$

DPP of order 3: (2), (3), (3 1), (3, 2), (3, 3), (3, 3; 2), (ϕ)

G. Andrews: $1, 2, 7, 42, 429, 7436, \ldots$

Totally symmetric self-complementary plane partitions



- an open problem:
  to find a natural bijection between ASMs of order n and TSSCPP of order n

Permutation pattern and ASM

We extend the notion of permutation pattern to ASMs.

Alternating sign matrix M contain permutation π if there is a submatrix D of M s.t. $d_{ij} = 1$ if $\pi(i) = j$; otherwise avoid, eg.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & - & 1 & 0 \\ 0 & 1 & - & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ contains } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & - & 1 & 0 \\ 0 & 1 & - & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ avoids } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Family of ASM avoiding (213) permutation

We study a family of ASMs that

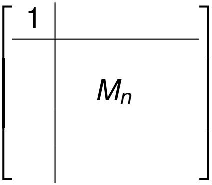
1. avoids the (213) permutation
2. the righmost 1 in row $i+1 \geq 2$ occurs to the right of the leftmost 1 in row i.

We prove that these matrices, denoted C_n have recursive nature!

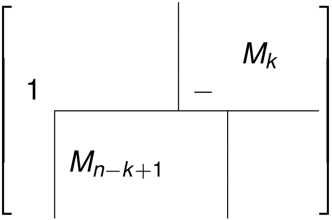
Recurence for our (213) family of ASMs

For $1x1$ size the only matrix is the identity matrix, otherwise we recognize two types of matrices

Type I



Type II



$$\begin{pmatrix}1\end{pmatrix} \quad \begin{pmatrix}1 & \\ & 1\end{pmatrix} \quad \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}, \begin{pmatrix} & 1 & \\ 1 & - & 1 \\ & 1 & \end{pmatrix}$$

$$\left[\begin{array}{c|ccc} 1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right], \left[\begin{array}{c|ccc} 1 & & & \\ \hline & & 1 & \\ & 1 & - & 1 \\ & & 1 & \end{array}\right],$$

$$\left[\begin{array}{cc|cc} & & 1 & \\ 1 & & - & 1 \\ \cline{2-4} & \multicolumn{1}{|c}{1} & \multicolumn{1}{c|}{} & \\ & \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{1} & \end{array}\right],$$

$$\left[\begin{array}{c|ccc} & 1 & & \\ & & 1 & \\ 1 & - & & 1 \\ \cline{2-4} & \multicolumn{1}{c|}{1} & & \end{array}\right], \left[\begin{array}{c|ccc} & & 1 & \\ & 1 & - & 1 \\ 1 & - & 1 & \\ \cline{2-4} & \multicolumn{1}{c|}{1} & & \end{array}\right].$$

Our recurrence keep both properties of the introduced ASM

- Type I obviously avoid (213)
- The same for Type II: there is no possibility that 1s in the first column form 2-1-3 relative positions

- Type I obviously respect the second constraint
- For Type II there are two situation to check: the 1s in the first column is not the righmost entry equal to 1 in its row, 1s at any postion in the $(k+1)$th row will be to the right of the leftmost 1s in the kth row

... and gives all such ASMs!

- $$\left[\begin{array}{c|c} 1 & \\ \hline & M_n \end{array}\right]$$

- $$\left[\begin{array}{c|ccc} M_1^0 & 1 & & \\ \cline{1-1} 1 & & - & 1 \\ \cline{3-4} & 1 & \multicolumn{2}{|c}{M_2^0} \end{array}\right]$$

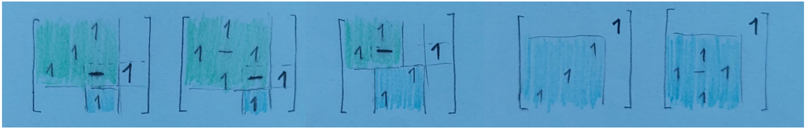
The Catalan convolution

- The number of matrices of order $n+1$ of type I is equal to m_n matrices of order n
- By the product rule, for the number of rest of matrices we have $\sum_{k=1}^{n-1} m_k m_{n-k+1}$

$$\begin{aligned} m_{n+1} &= m_n + \sum_{k=1}^{n-1} m_k m_{n-k+1} \\ &= \sum_{k=1}^{n} m_k m_{n-k+1} \end{aligned}$$

Further families

- ASMs avoiding permutation $\pi = (231)$ and meets the condition
  ii ') the leftmost 1 in the $i+1$ row is the left of the rightmost unit in the i-th row,
- ASMs avoiding permutation $\pi = (312)$ and satisfies condition ii '),
- ASMs avoiding permutation $\pi = (132)$ and satisfies condition ii).



Symmetries amoung families

$$\begin{bmatrix} & & & 1 & \\ 1 & & & - & 1 \\ & & 1 & & \\ & 1 & - & 1 & \\ & & 1 & & \end{bmatrix} \qquad \begin{bmatrix} & 1 & & & \\ 1 & - & & & 1 \\ & & 1 & & \\ & 1 & - & 1 & \\ & & 1 & & \end{bmatrix}$$

$$\begin{bmatrix} & & 1 & & \\ & 1 & - & 1 & \\ & & 1 & & \\ 1 & & & - & 1 \\ & & & 1 & \end{bmatrix} \qquad \begin{bmatrix} & & 1 & & \\ & 1 & - & 1 & \\ & & 1 & & \\ 1 & - & & & 1 \\ & 1 & & & \end{bmatrix}$$

Proposition

The vertically axially symmetric matrix M' of the matrix $M \in \mathcal{C}_n^1$ belongs to the family $\mathcal{C}_n^2$, horizontally axially symmetric matrix M'' belongs to the family $\mathcal{C}_n^3$ and centrally symmetric matrix M''' belongs to the family $\mathcal{C}_n^4$.

Proof:

From the definition of alternating sign matrices and symmetries above the defined ones, it follows that the matrices M', M'' and M''' are alternating sign matrices. If the matrix M' does not avoid the permutation $\pi = (231)$ then there are $m_{i'j} = m_{j'l} = m_{l'i} = 1$, such that $i < j < l$, $i' < j' < l'$ so we have the entriess of the matrix M such that $m_{i',n-j+1} = m_{j',n-l+1} = m_{l',n-i+1} = 1$, and $n - l + 1 < n - j + 1 < n - i + 1$, $i' < j' < l'$, so the matrix M has a permutation $\pi = (213)$.

This contradicts the definition of the matrix M. Thus, M' avoids the permutation $\pi = (231)$. In a similar way it can be proved that M'' and M''' avoid permutations $\pi = (312)$ and $\pi = (132)$ respectively. From the definition of matrix symmetry it follows that matrices M' and M'' satisfy property ii ') and also M''' satisfies property ii). □

Corollary

Matrices of family $\mathcal{C}_n^2$ are centrally symmetric to matrices of family $\mathcal{C}_n^3$ and horizontally symmetric to matrices of family $\mathcal{C}_n^4$, while matrices of families $\mathcal{C}_n^3$ and $\mathcal{C}_n^4$ are vertically symmetric to each other.

Refined enumeration I

$$B_{4,1} = 2,\ B_{4,2} = 1,\ B_{4,3} = 2$$

$$B = \left(\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 2 & & 1 & & 2 & & & & \\
 & & & 5 & & 2 & & 2 & & 5 & & & \\
 & & 14 & & 5 & & 4 & & 5 & & 14 & & \\
 & 42 & & 14 & & 10 & & 10 & & 14 & & 42 & \\
132 & & 42 & & 28 & & 25 & & 28 & & 42 & & 132 \\
 & & & & & & \vdots & & & & & &
\end{array}\right)$$

Refined enumeration I

Theorem

For the number $B_{n,k}$ of $\mathcal{C}$-matrices of order n having 1s in the first row and k-th column we have

$$B_{n,k} = \frac{1}{k(n-k)} \binom{2k-2}{k-1} \binom{2n-2k-2}{n-k-1}, \tag{1}$$

for $k < n$.

Refined enumeration I

Proof:

When $1 < k < n$, we argue by induction on the order n of matrix M. Let us assume that for every $m < n$ the number of matrices of order m having 1s in the first row and k-th column is equal to

$$C_{k-1}C_{m-k-1}.$$

$$C_{k-(n-r)-1}C_{r-(k-n+r)-1} = C_{k-(n-r)-1}C_{n-k-1}.$$

$$\begin{aligned}
B_{n,k} &= \sum_{r=n-k+1}^{n-1} C_{k-(n-r)-1}C_{n-k-1}C_{n-r-1} \\
&= C_{n-k-1}\sum_{r=n-k+1}^{n-1} C_{k-(n-r)-1}C_{n-r-1} \\
&= C_{n-k-1}\sum_{r=0}^{k-2} C_r C_{k-2-r} \\
&= C_{n-k-1}C_{k-1},
\end{aligned}$$

Refined enumeration II

Theorem

The number of $\mathcal{C}$-matrices of order n that have k entries -1 is equal to the Narayana number $N(n-1, k+1)$,

$$B_{n,k}^{(-)} = \frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1},$$

where $n \geq 2$ and $k = 0, 1, \ldots, n-2$.

Refined enumeration II

$$N(n,k) = N(n-1,k) + \sum_{i=1}^{n-1}\sum_{j=1}^{k-1} N(i,j)N(n-i-1,k-j) \qquad (2)$$

for $n > 0$, $1 \le k \le n$, with initial values $N(n,k) = 0$ for $k > n$, $n \neq 0$ and $N(0,1) = N(n,1) = N(n,n) = 1$ (M. Zabrocki, 2004, see OEIS).
The number of -1s in a $\mathcal{C}$-matrix of order n is at most $n-1$, i.e. $0 \le k < n-1$. For a matrix of order n obtained by the recurrence of type I, the number of -1s is equal to the number of -1s of the matrix of order $n-1$ from which it is obtained. For a matrix obtained by the recurrence of type II, the number of -1s is increased by 1 of the total number of -1s of both matrices from which it was obtained. It follows that the number $B^{(-)}_{n,k}$ of $\mathcal{C}$-matrices of order n with k -1s is obtained by summing

- the number of all matrices of order $n-1$ with k entries equal to -1, and
- the sum of the products of the number of all matrices of order i with j entries equal to -1 with the number of all matrices of order $n-i$ with $k-1-j$ entries equal to -1, $i = 2, \ldots, n-1$, $j = 0, \ldots, k-1$, that is

$$\sum_{i=2}^{n-1}\sum_{j=0}^{k-1} B^{(-)}_{i,j} B^{(-)}_{n-i,k-1-j}.$$

Refined enumeration II

$$B^{(-)}_{n,k} \;=\; B^{(-)}_{n-1,k} + \sum_{i=2}^{n-1}\sum_{j=0}^{k-1} B^{(-)}_{i,j} B^{(-)}_{n-i,k-1-j}.$$

Since it follows from (2) that

$$\begin{aligned} N(n-1,k+1) \;&=\; N(n-2,k+1) \\ &\quad + \sum_{i=2}^{n-1}\sum_{j=0}^{k-1} N(i-1,j+1)N(n-i-1,k-j), \end{aligned}$$

we have

$$B^{(-)}_{n.k} = N(n-1,k+1),$$

where $n \geq 2$, $0 \leq k \leq n-2$. This completes the proof.

The family of $\mathcal{F}$-matrices

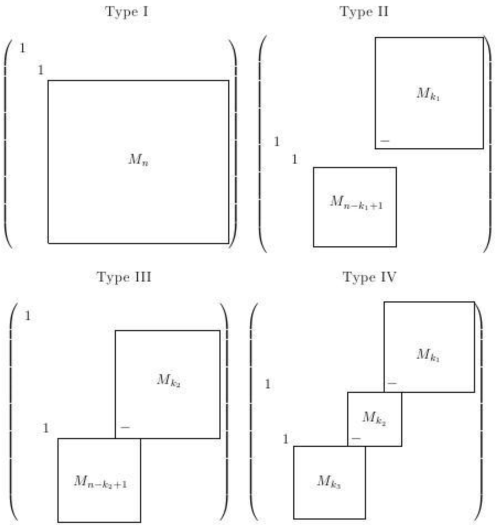
Definition

We let $\mathcal{F}_n$ denote a family of alternating sign matrices of an odd order $n \geq 1$ that

- avoid the permutation $\pi = (213)$,
- the rightmost 1 in a row $i + 1 \geq 2$ of a matrix occurs to the right of the leftmost 1s in row i, and
- if the j-th column of a matrix possesses the southeast 1s than $1 \equiv j \pmod 2$,

where $i = 1, \ldots, n-1$, $j = 1, \ldots, n$. Matrices from the set $\mathcal{F}_n$ we shall call $\mathcal{F}$-matrices.

Four types of $\mathcal{F}$-matrices



Enumeration of $\mathcal{F}$-matrices

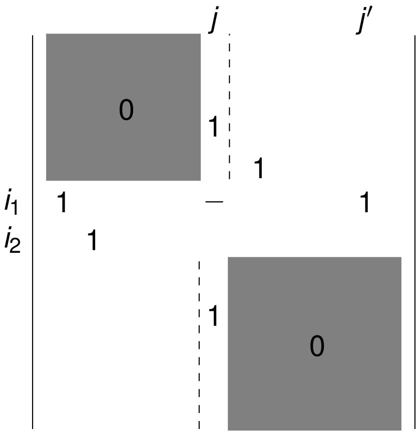
Theorem

For a non-negative integer r an alternating sign matrix M of order $n+2$, $n=2r+1$, is an $\mathcal{F}$-matrix if and only if it is formed recursively: for order 1 the only matrix is identity matrix, otherwise M is one of the types I, II, III or IV.

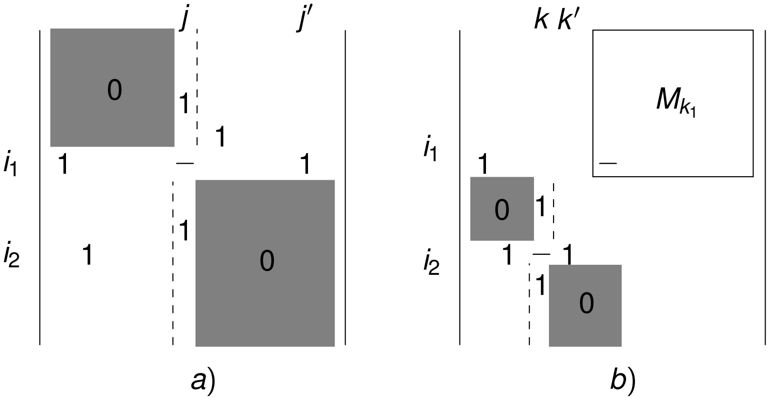
There are four different cases to consider,

- $i_1 = 1$, $i_2 = 2$,
- $i_1 > 1$, $i_2 = i_1 + 1$,
- $i_1 = 1$, $i_2 > 2$,
- $i_1 > 1$, $i_2 > i_1 + 1$,

Constructing $\mathcal{F}$-matrix of type II



Constructing $\mathcal{F}$-matrix of type IV



$$\begin{pmatrix} \mathbf{1}&&&&&& \\ &\mathbf{1}&&&&& \\ &&1&&&& \\ &&&1&&& \\ &&&&1&& \\ &&&&&1& \\ &&&&&&1 \end{pmatrix} \begin{pmatrix} \mathbf{1}&&&&&& \\ &\mathbf{1}&&&&& \\ &&&&1&& \\ &&&&&1& \\ &&1&&-&&1 \\ &&&1&&& \\ &&&&1&& \end{pmatrix} \begin{pmatrix} \mathbf{1}&&&&&& \\ &\mathbf{1}&&&&& \\ &&1&&&& \\ &&&&1&& \\ &&&&&1& \\ &&&1&-&&1 \\ &&&&1&& \end{pmatrix} \begin{pmatrix} &&&&1&& \\ &&&&&1& \\ \mathbf{1}&&&&-&&1 \\ &\mathbf{1}&&&&& \\ &&1&&&& \\ &&&1&&& \\ &&&&1&& \end{pmatrix}$$

$$\begin{pmatrix} &&1&&&& \\ &&&1&&& \\ &&&&1&& \\ &&&&&1& \\ \mathbf{1}&&-&&&&1 \\ &\mathbf{1}&&&&& \\ &&1&&&& \end{pmatrix} \begin{pmatrix} &&&&1&& \\ &&&&&1& \\ &&1&&-&&1 \\ &&&1&&& \\ \mathbf{1}&&-&&1&& \\ &\mathbf{1}&&&&& \\ &&1&&&& \end{pmatrix} \begin{pmatrix} &&1&&&& \\ &&&&1&& \\ &&&&&1& \\ &&&1&-&&1 \\ \mathbf{1}&&-&&1&& \\ &\mathbf{1}&&&&& \\ &&1&&&& \end{pmatrix} \begin{pmatrix} \mathbf{1}&&&&&& \\ &&&&1&& \\ &&&&&1& \\ &\mathbf{1}&&&-&&1 \\ &&1&&&& \\ &&&1&&& \\ &&&&1&& \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1}&&&&&& \\ &&1&&&& \\ &&&1&&& \\ &&&&1&& \\ &&&&&1& \\ &\mathbf{1}&-&&&&1 \\ &&1&&&& \end{pmatrix} \begin{pmatrix} \mathbf{1}&&&&&& \\ &&&&1&& \\ &&&&&1& \\ &&1&&-&&1 \\ &&&1&&& \\ &\mathbf{1}&-&&1&& \\ &&1&&&& \end{pmatrix} \begin{pmatrix} \mathbf{1}&&&&&& \\ &&1&&&& \\ &&&&1&& \\ &&&&&1& \\ &&&1&-&&1 \\ &\mathbf{1}&-&&1&& \\ &&1&&&& \end{pmatrix} \begin{pmatrix} &&&&1&& \\ &&&&&1& \\ \mathbf{1}&&&&-&&1 \\ &&1&&&& \\ &&&1&&& \\ &\mathbf{1}&-&&1&& \\ &&1&&&& \end{pmatrix}$$

Thank you for your attention!