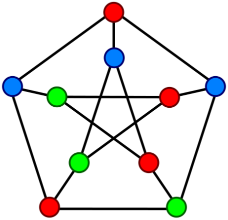
Harmonious coloring of the incidence graph of a design

Francesca Merola
Università Roma Tre

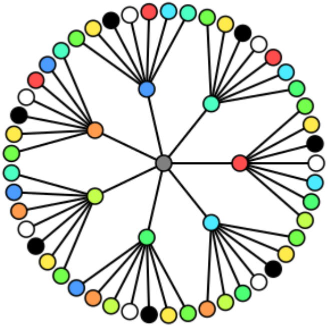
RICCOTA, Rijeka - July 2023

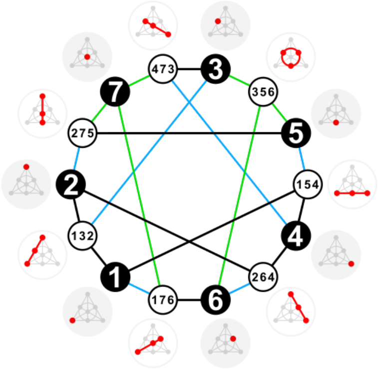
Joint work with
Marco Buratti
Giuseppe Mazzuoccolo
Anamari Nakić
Christian Rubio-Montiel
Tommaso Traetta
...



- color the vertices of a graph $G = (V, E)$ so that adjacent vertices have different colors
- the minimum number of colors needed is called the chromatic number of G, and denoted by $\chi(G)$
- many variations of this basic notion of coloring

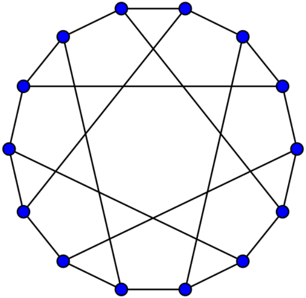
- a coloring is harmonious if each pair of colors appears on at most one pair of adjacent vertices
- the minimum number of colors needed is called the harmonious chromatic number of G, and denoted by $h(G)$
- clearly $\chi(G) \leq h(G) \leq |V| = n$
- and $|E| = m \leq \binom{h(G)}{2}$
- $\mathrm{diam}(G) \leq 2 \Rightarrow h(G) = n$





- $G_{\mathcal{D}}$ of a 2-$(v, k, 1)$ design $\mathcal{D}$ (or any incidence structure)
- is a bipartite graph with
  - one vertex per point
  - one vertex per block
  - an edge for any incident point block pair
  - $n = v + b$, $m = bk$

Levi graphs



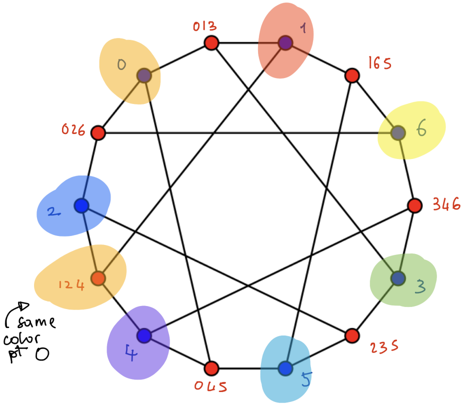
- the Levi graph or incidence graph $G_{\mathcal{D}}$ of a 2-$(v, k, 1)$ design (or any incidence structure)
- is a bipartite graph with
  - one vertex per point
  - one vertex per block
  - an edge for any incident point block pair
  - $n = v + b$, $m = bk$
- the Heawood graph is the Levi graph of the Fano plane
- it is the $(3, 6)$-cage
- the Levi graph of $PG(2, q)$ is the $(q + 1, 6)$-cage

harmonious colorings of Levi graphs of designs

- a recent preprint[1] considers the problem of calculating the HCN of Levi graphs
- if $\mathcal{D}$ is a 2-$(v, k, 1)$ design, $h(G_{\mathcal{D}}) \geq v$
- for which designs is this lower bound attained?
- for brevity, we call a design a Banff design if $h(G_{\mathcal{D}}) = v$

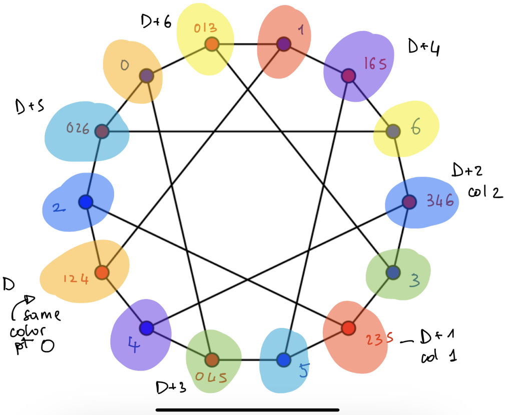
[1] Araujo-Pardo, Monellano-Ballestreros, Olsen, Rubio-Montiel, On the harmonious chromatic number of graphs, `arXiv:2206.04822`

- in the same preprint it is shown that when $\mathcal{D}$ is a projective plane $PG(2, q)$ $v \leq h(G) \leq v + 1$ $(v = q^2 + q + 1)$
- using difference methods we showed that $h(G) = v$
- the points "are" elements of $\mathbb{Z}_v$ - point i has color i.
- let $D = \{d_0, d_1, \ldots, d_q\}$ be a difference set for $PG(2, q)$
- choose a color $c \in \mathbb{Z}_v$ for D s.t. the list $\{\pm(c - d_i)\ i = 0, \ldots, q\}$ consists of all distinct elements
- D has color c and $D + i$ has color $c + i$
- a counting argument shows that the choice of a suitable c is always possible
- all the $PG(2, q)$ are Banff designs
- by suitably translating, we can choose the starting Difference Set D so that D is colored with $c = 0$



- eg $PG(2,2)$ is the $\mathbb{Z}_7$-development of the Difference Set $D = \{1,2,4\}$
- the set of points is the set of colors is $\mathbb{Z}_7$
- D has color 0

$PG(2,2)$



- eg $PG(2,2)$ is the $\mathbb{Z}_7$-development of the Difference Set $D = \{1,2,4\}$
- the set of points is the set of colors is $\mathbb{Z}_7$
- D has color 0
- $D + i$ has color i

Symmetric difference families

- a cyclic $(v, k, 1)$-Difference Family is a set of k-subsets $\{B_1, B_2 \ldots, B_s\}$ of $\mathbb{Z}_v$ s.t. $\Delta B_1 \cup \Delta B_2 \cup \cdots \cup \Delta B_s = \mathbb{Z}_v \setminus \{0\}$
- developing the blocks of the DF gives a cyclic $(v, k, 1)$-design
- assume that a DF has the additional property that for any x in any block of the DF,
  - x appears only once (ie the DF is disjoint)
  - $-x \pmod{v}$ does not belong to any block of the DF
- call such a DF symmetric
- Example 1: $\{\{7, 8, 11\}, \{4, 10, 12\}\}$ is a symmetric DF in $\mathbb{Z}_{13}$
- Example 2: let $p = 4k - 1$ be a prime, then the Paley $(4k - 1, 2k - 1, k - 1)$-DS consisting of the non-zero squares of $\mathbb{Z}_p$ is symmetric by construction (here we have $\lambda \geq 1$)

- if a 2-$(v, k, 1)$-design $\mathcal{D}$ is the development of a symmetric DF, then $h(G_{\mathcal{D}}) = v$ so $\mathcal{D}$ is a Banff design

using a result by Buratti and Pasotti[2], we were able to prove

Theorem

Let q be a prime power, $q \equiv 1 \pmod{k(k-1)}$. Then there exists a symmetric $(q, k, 1)$-DF, and thus a 2-$(q, k, 1)$ Banff design, for all $q > q_0$.

[2] Buratti, Pasotti, Combinatorial designs and the theorem of Weil on multiplicative character sums, Finite Fields Appl. 15 (2009), 332–344

- Novák's conjecture (1974) states that any cyclic STS(v), $v \equiv 1 \pmod 6$ can be obtained via a disjoint difference family
- significant recent progress[3] on the truth of the conjecture, and its generalization to cyclic 2-$(v, k, 1)$-designs
- there exists a strong version of Novák's conjecture
- stating that any cyclic STS(v), $v \equiv 1 \pmod 6$ can be obtained via a symmetric difference family
- not much is known on the truth of this conjecture
- if true, then any cyclic STS(v) is a Banff design

[3] Feng, Horsley, Wang, Novák's conjecure on cyclic Steiner triple systems and its genralizations, JCT A **184** (2021), # 105515

HCN of STSs and nestings

- we proved the existence of Banff STS(v), $v \equiv 1 \pmod 6$ and showed that v colors are not enough when $v \equiv 3 \pmod 6$
- then realised this already known, when reformulated in terms of nestings
- a STS(v) can be nested if it is possible to add a point to each triple so that the result is a $(v, 4, 2)$-design
- nested STS(v) exist iff $v \equiv 1 \pmod 6$ (Colbourn and Colbourn 1983, Stinson 1985)
- a nested STS(v) has HCN equal to v