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On Flag-Transitive Symmetric 2-Designs Arising from Cameron–Praeger Construction

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(Joint Work with Cheryl E. PRAEGER)

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- **non-trivial** if $2 < k < v - 1$.
- **symmetric** if $|\mathcal{B}| = v$ or, equivalently, $r = k$, where $r = \frac{(v-1)\lambda}{k-1}$

Preliminaries

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A resolvable 2-design $\mathcal{D}$ in which blocks in different classes have the same number of points in common is called **affine resolvable**.

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- G acts **flag-transitively** on $\mathcal{D}$ if for any flags (x, B) and (x', B') of $\mathcal{D}$ there is $\gamma \in G$ such that $(x, B)^{\gamma} = (x', B')$.

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1. A resolvable 2-(v_0, k_0, λ_0) design $\mathcal{D}_0 = (\Delta_0, \mathcal{L}_0)$ with r_0 parallel classes $\mathcal{P}_0 = \{P_1, ..., P_{r_0}\}$ parallel classes and each class consists of s_0 blocks.

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2. A symmetric 2-(v_1, r_0, λ_1) design $\mathcal{D}_1 = (\Delta_1, \mathcal{L}_1)$ together with $(\psi_\beta)_{\beta \in \mathcal{L}_1}$, where $\psi_\beta : \mathcal{P}_0 \to \beta$ is a bijection for each $\beta \in \mathcal{L}_1$ such that $(P, i) \in \mathcal{P}_0 \times \Delta_1$ there is a unique $\beta \in \mathcal{L}_1$ such that $\psi_\beta(P) = i$.

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3. A transversal design $\mathcal{D}_2 = (\Delta_2, \mathcal{L}_2)$ whose point set is partitioned in r_0 groups each of size s_0, and Δ_2 is identified with $\cup_{i=1}^{r_0} P_i$; each block has size $k_2 \leq r_0$ and meets each group in at most one point; and each two points in different groups lie in exactly λ_2 blocks.

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It follows from the definition of Δ_2 that, for each $\gamma \in \mathcal{L}_2$, $\beta \in \mathcal{L}_1$ and $j \in \beta$ either $\gamma \cap \psi_\beta^{-1}(j) = \varnothing$ or $\gamma \cap \psi_\beta^{-1}(j)$ is a single block of the parallel class $\psi_\beta^{-1}(j)$.

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Let $\mathcal{D} = (\Delta_0 \times \Delta_1, \mathcal{B})$ be the incidence structure, where $\mathcal{B} = \cup_{\beta \in \mathcal{L}_1} \mathcal{B}_\beta$, $\mathcal{B}_\beta = \{B_\beta(\gamma) : \gamma \in \mathcal{L}_2\}$ and

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- $\mathcal{D}$ is a 2-design if and only if $\lambda_1 = \lambda_0 \frac{(r_0 - 1)s_0}{k_2 - 1}$, and in this case is a 2-$(v_0 v_1, k_0 k_2, \lambda_1 \lambda_2)$ design.

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- $\mathcal{D}$ is symmetric if and only if $r_0(r_0 - 1)s_0\lambda_2 = k_0 k_2(k_2 - 1)$.

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Theorem [Cameron-Praeger (2016)]

If there exists an affine resolvable 2-$\left(s_0^2\mu, s_0\mu, \frac{s_0\mu-1}{s_0-1}\right)$ design with $r_0 = \frac{s_0^2\mu-1}{s_0-1}$ parallel classes, in which blocks belonging to distinct classes intersects in exactly μ points, then there is a 2-$\left(s_0^2\mu(r_0+1), s_0\mu r_0, \mu(r_0-1)\right)$ design.

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Definition

Any (symmetric) 2-design isomorphic to one arising from the Cameron-Praeger construction will be called of **CP-type**.

An example with $s_0 = 3$ and $\mu = 1$

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The input is:

1. $\mathcal{D}_0 = (\Delta_0, \mathcal{L}_0) \cong AG_2(3)$. Here $\Delta_0 = \{1, ..., 9\}$ and the resolution of $\mathcal{L}_0$ is $\{P_1, P_2, P_3, P_4\}$ with

$$P_1 = \{\{1,2,4\},\{3,5,7\},\{6,8,9\}\}$$
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2. $\mathcal{D}_1 = (\Delta_1, \mathcal{L}_1)$ is the trivial 2-$(5,4,3)$ symmetric design. Here, $\Delta_1 = \{1,2,3,4,5\}$ and $\mathcal{L}_1 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$, where

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3. $\mathcal{D}_2$ is the dual of $\mathcal{D}_0$ and an exemplary block is

$$\gamma_0 = \{\{1,2,4\},\{1,3,6\},\{1,5,8\},\{1,7,9\}\}$$

4 A Latin square of order 5:

	P_1	P_2	P_3	P_4	∞
β_1	1	2	3	4	5
β_2	2	3	4	5	1
β_3	3	4	5	1	2
β_4	4	5	1	2	3
β_5	5	1	2	3	4

The output is a 2-$(45, 12, 3)$ design with $\{1, .., 9\} \times \{1, .., 5\}$ as a point set and with an exemplary block determined below:

- $\beta_2 = \{1, 2, 3, 5\}$;
- $\psi_{\beta_2}(P_1) = 2 \quad \psi_{\beta_2}(P_2) = 3 \quad \psi_{\beta_2}(P_3) = 4 \quad \psi_{\beta_2}(P_4) = 5$
- $\gamma_0 = \{\{1, 2, 4\}, \{1, 3, 6\}, \{1, 5, 8\}, \{1, 7, 9\}\}$
$\gamma_0 \cap \psi^{-1}_{\beta_2}(1) = \emptyset$	$\gamma_0 \cap \psi^{-1}_{\beta_2}(2) = \{1, 2, 4\}$
$\gamma_0 \cap \psi^{-1}_{\beta_2}(3) = \{1, 3, 6\}$	$\gamma_0 \cap \psi^{-1}_{\beta_2}(5) = \{1, 7, 9\}$
- $B_{\beta_2}(\gamma_0) = (\{1, 2, 4\} \times \{2\}) \cup (\{1, 3, 6\} \times \{3\}) \cup (\{1, 7, 9\} \times \{5\})$.

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2. $\mathcal{D}$ is the 2-$(2^{2n}, 2^{n-1}(2^n-1), 2^{n-1}(2^{n-1}-1))$ design $S^{-}(n)$ with $n \geq 2$ described in Cameron-Seidel (1973), and $G \cong 2^{2n} : GL_2(n)$.

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3. $\mathcal{D}$ is a 2-$(1408, 336, 80)$ design constructed by Cameron-Praeger (2016) and $G \cong 2^6 : ((3 \cdot M_{22}) : 2)$.

Invariants

Let G be a flag-transitive automorphism of a non-trivial 2-(v, k, λ) design $\mathcal{D}$ and let Σ be a non-trivial G-invariant partition of the point set of $\mathcal{D}$ in d classes each of size c.

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- $|\Delta \cap B| \in \{0, \ell\}$ for each $\Delta \in \Sigma$ and each block B of $\mathcal{D}$ (Praeger-Zhou 2006);

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(1) $\mathcal{D}_\Delta = (\Delta, \mathcal{B}_\Delta)$, where $\mathcal{B}_\Delta = \{B \in \mathcal{B} : B \cap \Delta \neq \varnothing\}$, is a 2-$(c, \ell, \lambda/\theta)$ design;

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(1) $\mathcal{D}_\Delta = (\Delta, \mathcal{B}_\Delta)$, where $\mathcal{B}_\Delta = \{B \in \mathcal{B} : B \cap \Delta \neq \varnothing\}$, is a 2-$(c, \ell, \lambda/\theta)$ design;

(2) G_Δ^Δ acts flag-transitively on $\mathcal{D}_\Delta$.

Invariants

Let G be a flag-transitive automorphism of a non-trivial 2-(v, k, λ) design $\mathcal{D}$ and let Σ be a non-trivial G-invariant partition of the point set of $\mathcal{D}$ in d classes each of size c. The following are invariant:

- $|\Delta \cap B| \in \{0, \ell\}$ for each $\Delta \in \Sigma$ and each block B of $\mathcal{D}$ (Praeger-Zhou 2006);
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Problem

Determine $(\mathcal{D}, G)$ when $\mathcal{D}$ is a symmetric 2-design of CP-type with $\mathcal{D}_\Delta$ affine resolvable.

More information on the previous examples

More information on the previous examples

Examples

1. $\mathcal{D}$ is one the four 2-$(96, 20, 4)$ designs constructed by Law-Praeger-Reichard (2007). Here, $\mathcal{D}_0 \cong \mathcal{D}_\Delta \cong AG_2(4)$.
2. $\mathcal{D}$ is the 2-$(2^{2n}, 2^{n-1}(2^n - 1), 2^{n-1}(2^{n-1} - 1))$ design $S^-(n)$ with $n \geq 2$ described in Cameron-Seidel (1973), and $G \cong 2^{2n} : GL_2(n)$. Here, $\mathcal{D}_0 \cong \mathcal{D}_\Delta \cong AG_n(2)$.
3. $\mathcal{D}$ is a 2-$(1408, 336, 80)$ design constructed by Cameron-Praeger (2016) and $G \cong 2^6 : ((3 \cdot M_{22}) : 2)$. Here, $\mathcal{D}_0 \cong \mathcal{D}_\Delta \cong AG_3(4)$.

The family $\mathcal{F}$

Definition

Let $\mathcal{D}$ be a non-trivial symmetric 2-design admitting a flag-transitive point-imprimitive automorphism group G.
Then $(\mathcal{D}, G) \in \mathcal{F}$ if the following hold:

1. $\mathcal{D}_\Delta$ is an affine resolvable 2-$\left(s_0^2\mu, s_0\mu, \frac{s_0\mu-1}{s_0-1}\right)$ design with $r_0 = \frac{s_0^2\mu-1}{s_0-1}$ parallel classes, in which blocks belonging to distinct classes intersects in exactly μ points;
2. $\mathcal{D}$ is a 2-$\left(s_0^2\mu(r_0+1), s_0\mu r_0, \mu(r_0-1)\right)$ design.

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If $\mathcal{D}$ is a 2-design of CP-type admitting a flag-transitive point-imprimitive automorphism group G with $\mathcal{D}_\Delta$ affine resolvable, then $(\mathcal{D}, G) \in \mathcal{F}$.

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Lemma [Alavi, Daneshkhah, M., Zhou et al. (2022)]

The group G_Δ^Δ is flag-transitive and point primitive on $\mathcal{D}_\Delta$, and one of the following holds:

(I) G_Δ^Δ **is almost simple** and one of the following holds:
  (a) $\mathcal{D}_\Delta$ is a 2-$(8,4,3)$ design with $G_\Delta^\Delta = PSL_2(7)$;
  (b) $\mathcal{D}_\Delta$ is a 2-$(12,6,5)$ design with $G_\Delta^\Delta = M_{11}$;

(II) G_Δ^Δ **is of affine type** and $\mathcal{D}_\Delta$ is a 2-(p^i, p^j, λ_0) design with either $\lambda_0 = 1$ or $\lambda_0 = \frac{p^j-1}{p^{\gcd(j,i/z)}-1}$ for some $z \mid i$ such that $\gcd(j,z,i/z)=1$, or $\lambda_0 = \frac{p^j-1}{a}$ for some $a \mid p^{\gcd(j,i)}-1$. The points and blocks of $\mathcal{D}_\Delta$ are the points and (certain) j-subspaces of $AG_i(p)$.

Theorem [M., Praeger (2023+)]

Let $\mathcal{D}$ be a non-trivial symmetric 2-design admitting a flag-transitive point-imprimitive automorphism group G. If $(\mathcal{D}, G) \in \mathcal{F}$ and G_Δ^Δ is of affine type,

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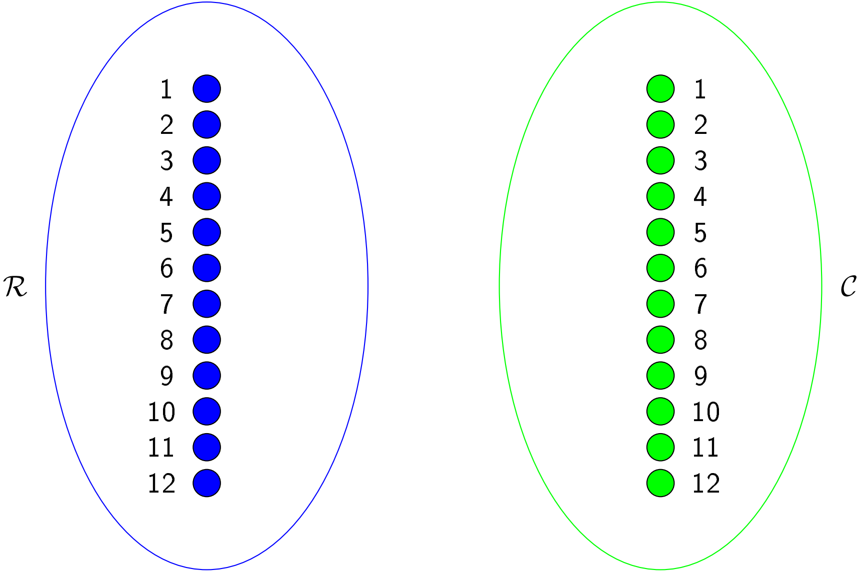
The FT+PI example involving M_{12}



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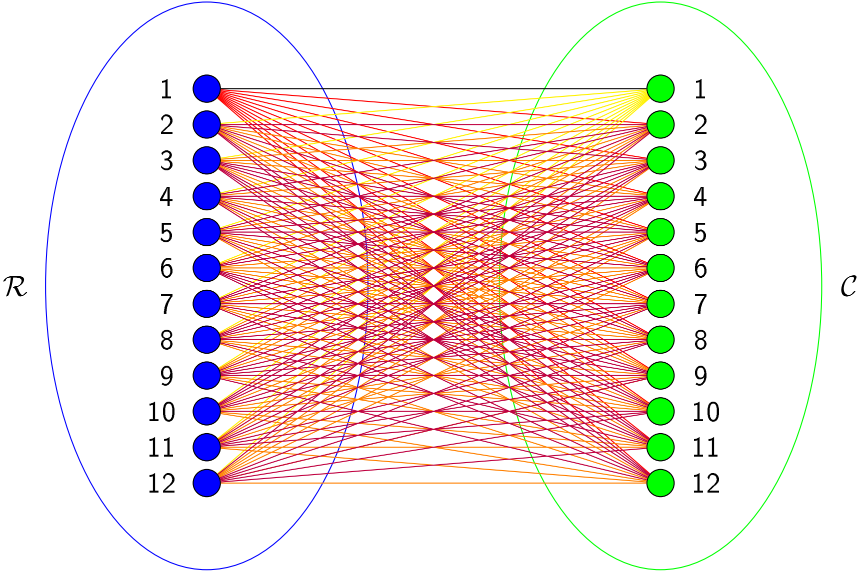

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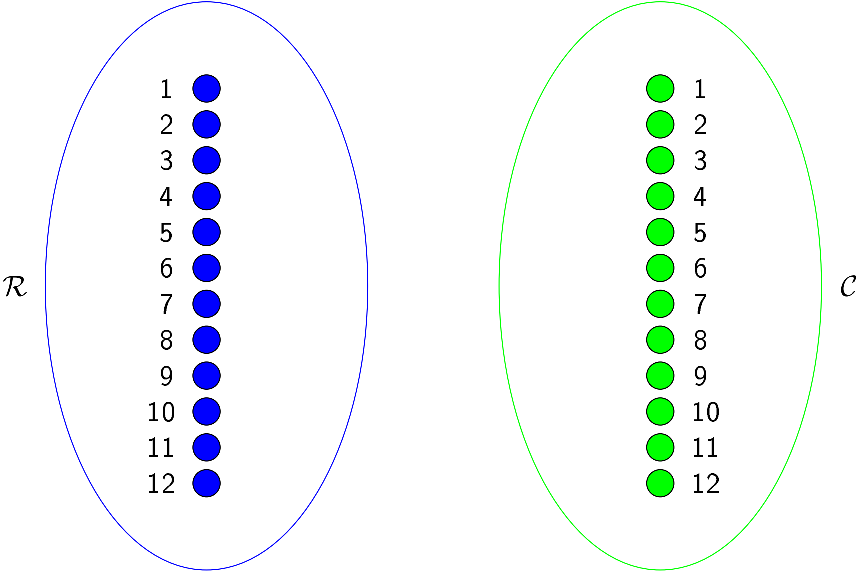

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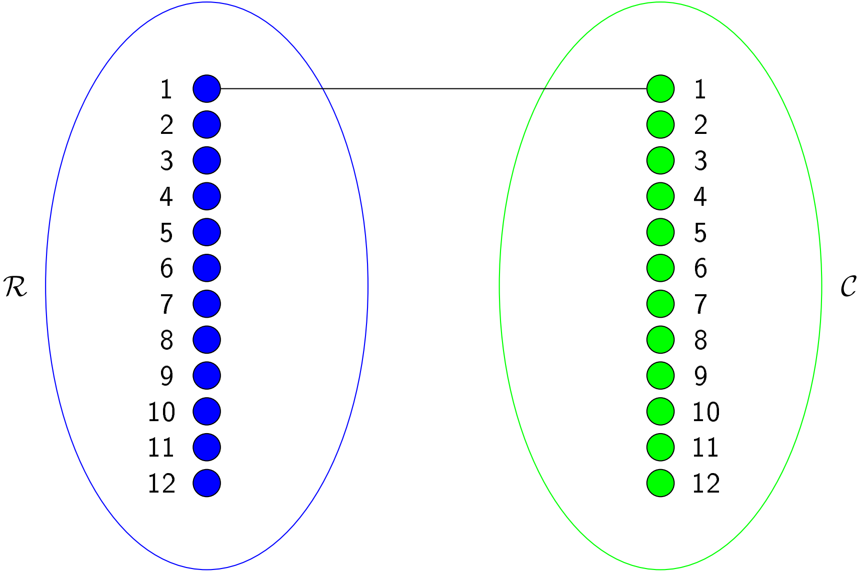

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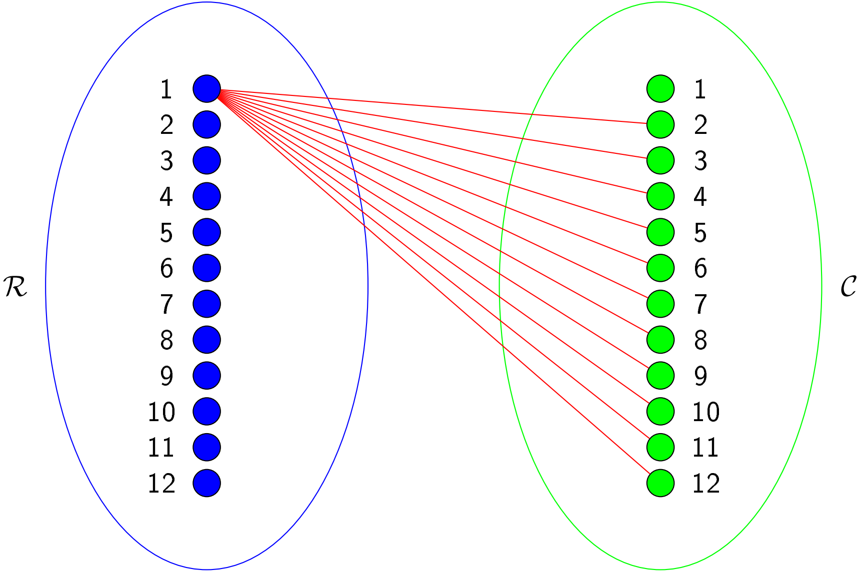

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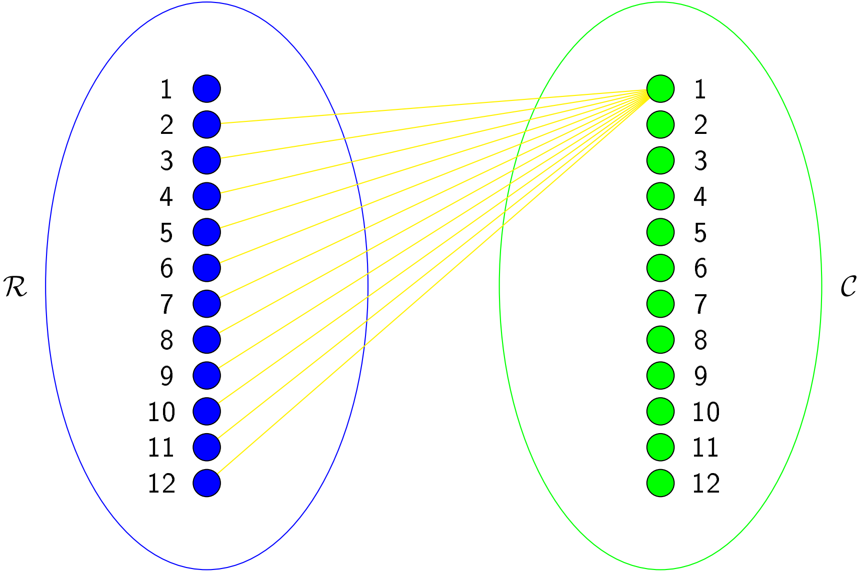

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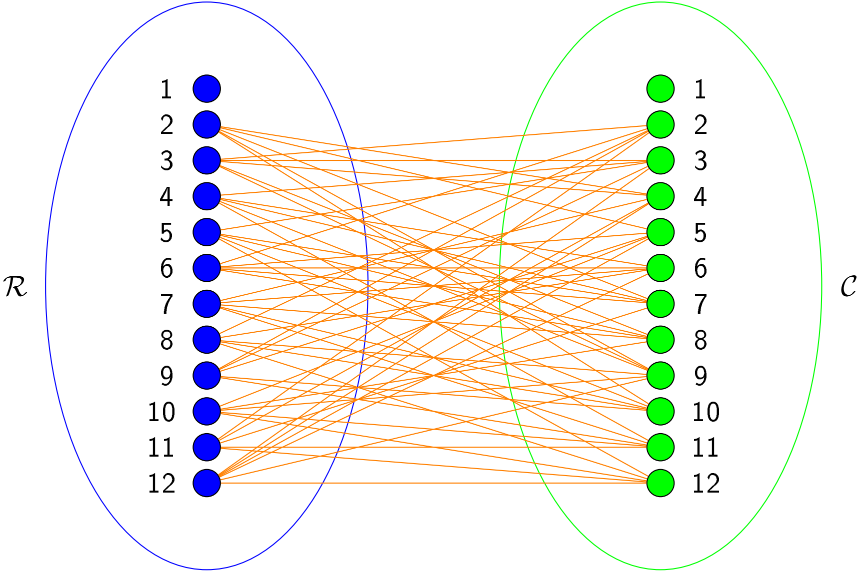

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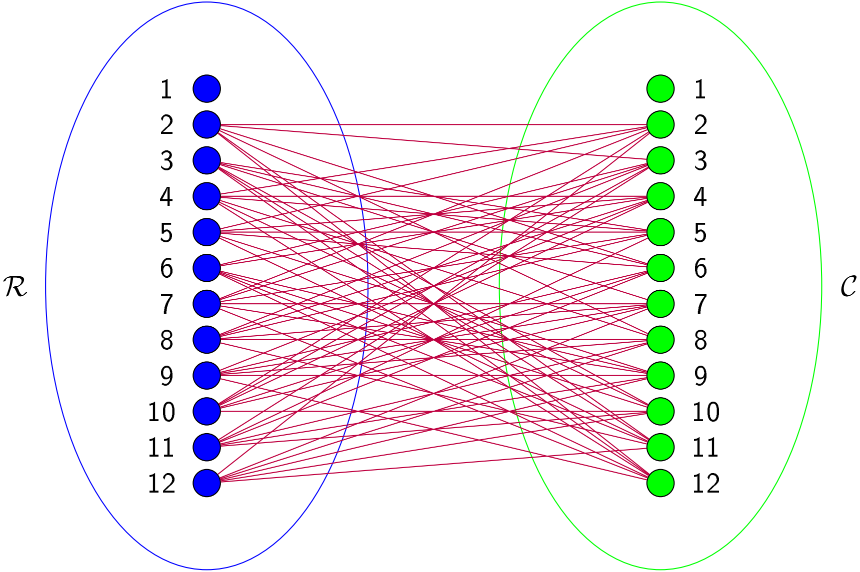

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- Γ is the incidence graph of the complementary design of the 2-$(11, 5, 2)$ Paley-Hadamard design.

Hvala Na Pažnji!