Distance-regular graphs with classical parameters that support a uniform structure: case $q \geq 2$

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Notations and preliminaries

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$\Gamma = (X, \mathcal{R})$: (simple, connected, undirected) graph with vertex set X and edge set $\mathcal{R}$

$D = \max\{\partial(x, y) \mid x, y \in X\}$, with ∂ distance on Γ : *diameter* of Γ

V : vector space over $\mathbb{C}$ of column vectors with coordinates indexed by X and entries in $\mathbb{C}$

$\varepsilon := \varepsilon(x) = \max\{\partial(x, y) \mid y \in X\}$: *eccentricity* of x (fixed)

E_i^* : i-th *dual idempotent* of Γ w.r.t. x $(0 \leq i \leq \varepsilon)$

$T := T(x)$: *Terwilliger algebra* of Γ w.r.t. x

L, F, and R : *lowering*, *flat*, and *raising* matrices w.r.t. x

$\Rightarrow$ $T = \langle L, F, R, \{E_i^*\}_{i=0}^{\varepsilon} \rangle$

$W \leq V$: (*irreducible*) T-module (with *endpoint* r, *dual endpoint* t, and *diameter* d)

W is ***thin*** if $\dim(E_i^* W) \leq 1$ $(0 \leq i \leq \varepsilon)$.

Γ is ***r-thin*** if every irreducible T-module with endpoint r is thin.

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Assume Γ non-bipartite.

Γ_f : (bipartite and connected) subgraph of Γ with $F = 0$

$T_f := T_f(x)$: Terwilliger algebra of Γ_f w.r.t. x

$\Rightarrow T_f = \langle L, R, \{E_i^*\}_{i=0}^{\varepsilon} \rangle$

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$\Rightarrow$ $T_f = \langle L, R, \{E_i^*\}_{i=0}^{\varepsilon} \rangle$

Lemma

Let W denote a T-module. Then,

- W is a T_f-module.
- If W is a thin irreducible T-module, then W is a thin irreducible T_f-module.

Definition(s)

Consider a so-called *parameter matrix* $U = (e_{ij})_{1 \leq i,j \leq \varepsilon}$ over $\mathbb{C}$, i.e,

- $e_{ii} = 1$ $(1 \leq i \leq \varepsilon)$,
- $e_i^- := e_{i,i-1} \neq 0$ $(2 \leq i \leq \varepsilon)$ or $e_i^+ := e_{i-1,i} \neq 0$ $(2 \leq i \leq \varepsilon)$,
- $\det(e_{ij})_{s \leq i,j \leq t} \neq 0$ $(1 \leq s \leq t \leq \varepsilon)$.

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Γ *supports a uniform structure w.r.t. x* if Γ_f *admits a uniform structure* (U, f) (w.r.t. x), with $f = \{f_i\}_{i=1}^{\varepsilon}$, $f_i \in \mathbb{C}$, i.e.,

$$e_i^- RL^2 + LRL + e_i^+ L^2 R = f_i L \tag{1}$$

is satisfied on $E_i^* V$ $(1 \le i \le \varepsilon)$.

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is satisfied on $E_i^* V$ $(1 \leq i \leq \varepsilon)$.

Proposition

(1) holds on $E_i^* V$ if and only if (1) holds on $E_i^* W$ for every irreducible T-module W.

Two T-modules W and W' are *T-isomorphic* if there is a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)\, W = 0$ for all $B \in T$.

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Theorem (P. Terwilliger, 1990)

Let G denote a **bipartite** graph and fix $x \in V(G)$.
Let $T = T(x)$ denote the Terwilliger algebra of G, and assume G admits a **uniform structure** with respect to x. Then,

- Every irreducible T-module is thin.
- Let W denote an irreducible T-module with endpoint r and diameter d. Then, the isomorphism class of W is determined by r and d.

Further definitions

Γ is *distance-regular* if, for all integers $0 \le h, i, j \le D$ and all $z, y \in X$ with $\partial(z, y) = h$, the number

$$p^h_{ij} := |\Gamma_i(z) \cap \Gamma_j(y)|$$

is independent of the choice of z, y. The constants p^h_{ij} are the *intersection numbers* of Γ.

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- It is enough to consider $c_i := p^i_{1\,i-1}$ $(1 \leq i \leq D)$, $a_i := p^i_{1i}$ $(0 \leq i \leq D)$, $b_i := p^i_{1\,i+1}$ $(0 \leq i \leq D-1)$, where $c_0 := 0, b_D := 0$.
- $k_i := p^0_{ii} = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$ $(0 \leq i \leq D)$: *valencies* of Γ
- Γ is regular with valency $k := b_0 = k_1$ and $c_i + a_i + b_i = k$ $(0 \leq i \leq D)$.
- $\varepsilon(z) = D$ for every $z \in X$, and $\varepsilon = D$
- Γ is bipartite if and only if $a_i = 0$ for $0 \leq i \leq D$.

A *strongly regular* graph, i.e., $\mathrm{srg}(v, k, \lambda, \mu)$, is a regular graph with v vertices and valency k, such that every two (distinct) vertices have λ or μ common neighbors depending on whether the vertices are respectively nonadjacent or not.

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$\Rightarrow$ A *connected* strongly regular graph is a distance-regular of diameter 2.

Theorem

A regular graph has 3 distinct eigenvalues if and only if it is (connected) strongly regular.

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Theorem

A regular graph has 3 distinct eigenvalues if and only if it is (connected) strongly regular.

$\Rightarrow$ A *disconnected* strongly regular graph is a disjoint union of cliques of the same size.

Proposition

Let G denote a strongly regular graph. If G is a disjoint union of cliques, then -1 is an eigenvalue for G, and vice-versa.

Let Γ be a distance-regular graph with $D \geq 3$.

Γ has *classical parameters* (D, q, α, β) with $q \neq 1$ if

$$c_i = \frac{q^i - 1}{q - 1}\left(1 + \alpha \frac{q^{i-1} - 1}{q - 1}\right) \qquad (1 \leq i \leq D),$$

$$b_i = \frac{q^D - q^i}{q - 1}\left(\beta - \alpha \frac{q^i - 1}{q - 1}\right) \qquad (0 \leq i \leq D - 1),$$

where $q, \alpha, \beta \in \mathbb{C}$.

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where $q, \alpha, \beta \in \mathbb{C}$.

$\Rightarrow$ $q \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $\alpha, \beta \in \mathbb{Q}$.

$\Rightarrow$ Such a graph Γ is *Q-polynomial* (A. Brouwer et al., 1989).

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Assume that Γ is (non-bipartite) distance-regular, having classical parameters (D, q, α, β) with $D \geq 4$ and $q \geq 2$, and that Γ is 1-thin.

Local graph and thin irreducible T-modules

$k = b_0 = \theta_0 > \theta_1 > \cdots > \theta_D$: (distinct) *eigenvalues of* Γ

$\Delta := \Delta(x)$: subgraph of Γ induced on the set of vertices in X adjacent to x, known as the *local graph of* Γ w.r.t. x.

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- Δ has $k = b_0$ vertices and is regular with valency a_1.
- $a_1 = \eta_1 \geq \eta_2 \geq \cdots \geq \eta_k (\geq -a_1)$: eigenvalues of Δ, i.e, *local eigenvalues of* Γ w. r. t. x
- $\tilde{\theta_1} \leq \eta_i \leq \tilde{\theta_D}$ $(2 \leq i \leq k)$, with $\tilde{\theta_1} = -1 - b_1(1+\theta_1)^{-1}$ and $\tilde{\theta_D} = -1 - b_1(1+\theta_D)^{-1}$

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Definition

If W is any thin irreducible T-module with endpoint 1, then $E_1^* W$ is a one-dimensional eigenspace for $E_1^* A E_1^*$, whose eigenvalue η is called the *local eigenvalue of W*.

$\Rightarrow \eta \in \{\eta_2, \eta_3, \ldots, \eta_k\}$, so $\tilde{\theta_1} \leq \eta \leq \tilde{\theta_D}$.

Theorem (P. Terwilliger, 2002)

Let W denote a thin irreducible T-module with endpoint 1 and local eigenvalue η, and W' denote an irreducible T-module. Then, the followings are **equivalent**.

- W and W' are isomorphic as T-modules.
- W' is thin with endpoint 1 and local eigenvalue η.

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- W and W' are isomorphic as T-modules.
- W' is thin with endpoint 1 and local eigenvalue η.

Proposition (J. Go and P. Terwilliger, 2002)

Let W denote a thin irreducible T-module with endpoint 1, diameter d, and local eigenvalue η.
Then, the followings hold.

- If $\eta \in \{\tilde{\theta}_1, \tilde{\theta}_D\}$, then $d = D - 2$.
- If $\tilde{\theta}_1 < \eta < \tilde{\theta}_D$, then $d = D - 1$.

Theorem (P. Terwilliger, 2004)

Let Φ denote the set of distinct scalars among $\eta_2, \eta_3, \ldots, \eta_k$. For $\eta \in \Phi$, let m_η denote the number of times η appears among $\eta_2, \eta_3, \ldots, \eta_k$.

Then, there exist polynomials $p_0 = 1, p_1, \ldots, p_D$ (given by a known recursive formula) with real coefficients such that

$$1 + \sum_{\substack{\eta \in \Phi \\ \eta \neq -1}} \frac{p_{i-1}(\tilde{\eta})}{p_i(\tilde{\eta})(1+\tilde{\eta})} m_\eta \leq \frac{k}{b_i} \qquad (1 \leq i \leq D-1), \qquad (2)$$

where $\tilde{\eta} = -1 - b_1(1+\eta)^{-1}$.

Additionally, the **equality** in (2) for $1 \leq i \leq D-1$ holds if and only if every irreducible T-module with endpoint 1 is thin.

Our analysis: Γ supporting a uniform structure

$\Rightarrow$ The isomorphism class of a thin irreducible T-module W with endpoint 1 is determined by its local eigenvalue η.

For *our* Γ (1-thin, non-bipartite distance-regular with classical parameters (D, q, α, β), $D \geq 4$, $q \geq 2$), it is known that η is in the set

$$\left\{ \eta_1 := -q - 1,\ \eta_2 := \beta - \alpha - 1,\ \eta_3 := -1,\ \eta_4 := \alpha \frac{q^{D-1} - 1}{q - 1} - 1 \right\}$$

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- $\eta \in \{\eta_1, \eta_2\} \Rightarrow d = D-2$, otherwise, $d = D-1$.
- η_1, η_2, and η_3 are distinct, and $\eta_4 \neq \eta_1$.
- $\eta_4 = \eta_2 \iff \beta = \alpha \frac{q^D-1}{q-1}$
- $\eta_4 = \eta_3 \iff \alpha = 0$

Case $\alpha \neq 0$

Proposition 1 (B. Fernández, R. Maleki, Š. Miklavič, G.M.)

Let W and W' denote two non-isomorphic, thin irreducible T-modules with endpoint 1. Then, W and W' remain non-isomorphic when considered as T_f-modules.

Proposition 2 (B. Fernández, R. Maleki, Š. Miklavič, G.M.)

If Γ supports a uniform structure, then there are, up to isomorphism, exactly two thin irreducible T-modules with endpoint 1, one with diameter $D-2$ and the other with diameter $D-1$.

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$\Rightarrow$ Δ is not complete (otherwise $b_1 = 0$), and has at most three distinct eigenvalues, i.e., Δ is strongly regular.

$$\left\{\underset{D-2}{\eta_1} := -q-1,\ \underset{D-2}{\eta_2} := \beta-\alpha-1,\ \underset{D-1}{\eta_3} := -1,\ \underset{D-1}{\eta_4} := \alpha\frac{q^{D-1}-1}{q-1}-1\right\}$$

⇒ Pairs to be considered are those corresponding to <u>different</u> diameters.

⇒ The case $\{\eta_1, \eta_3\}$ never occurs since both eigenvalues would be negative.

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Lemma 1 (B. Fernández, R. Maleki, Š. Miklavič, G.M.)

Let Δ be the local graph of Γ with eigenvalues a_1, r, s with $a_1 \geq r \geq 0$ and $s < 0$. Then, $\{r, s\} \neq \{\eta_2, \eta_3\}$ and $\{r, s\} \neq \{\eta_2, \eta_4\}$.

Sketch of the proof

- $\{r,s\} = \{\eta_2,\eta_3\} \iff \Delta$ is a disjoint union of cliques ($\eta_3 = -1$) with $a_1 = \eta_2 = \beta - \alpha - 1 \iff \alpha = 0$: contradiction.
- $\{r,s\} = \{\eta_2,\eta_4\}$: the equality

$$1 + \sum_{\substack{\eta\in\Phi \\ \eta\neq -1}} \frac{p_{i-1}(\tilde{\eta})}{p_i(\tilde{\eta})(1+\tilde{\eta})} m_\eta = \frac{k}{b_i} \qquad (1 \leq i \leq D-1),$$

  holding for every 1-thin graph, is not satisfied in particular for $i = 1, D-1$; otherwise $\beta = 0$ when $r = \eta_2$ ($s = \eta_4$), and $s = \eta_2 = 0$ when $r = \eta_4$.

Only the case $\{r, s\} = \{\eta_1, \eta_4\}$ remains to be considered.

$\Rightarrow$ The previous equality is verified for every $1 \leq i \leq D-1$, and

$$\beta = \alpha \frac{q^{D-1}-1}{q-1} - q, \qquad \mu = \alpha(q+1).$$

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Theorem (Neumaier, 1979)

Let G be a strongly regular graph with parameters (n, k, λ, μ) and eigenvalues $k > r > s$. Then, at least one of the following conditions must hold:

(a) $r \leq \max\{2(-s-1)(\mu+1+s), s(s+1)(\mu+1)/2 - s - 1\}$.

(b) $\mu = s^2$: G is a Steiner graph derived from a Steiner 2-system in which each line contains s points.

(c) $\mu = s(s+1)$: G is a Latin square graph derived from an s-net.

Lemma 2 (B. Fernández, R. Maleki, Š. Miklavič, G.M.)

Let Δ be the local graph of Γ with eigenvalues a_1, r, s with $a_1 > r = \alpha\frac{q^{D-1}-1}{q-1} - 1$ and $s = -q-1$. Then, case (a) never happens.

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Let Δ be the local graph of Γ with eigenvalues a_1, r, s with $a_1 > r = \alpha\frac{q^{D-1}-1}{q-1} - 1$ and $s = -q - 1$. Then, case (a) never happens.

Sketch of the proof

- Claim 1: Δ is not a conference graph.
- Claim 2: $r \geq 1$.
- The integrality of λ yields that (a) cannot be.

Two feasible families

Cases $\mu = s^2$ and $\mu = s(s+1)$ are both feasible, and the classical parameters of the respective distance-regular graphs are

$$\left(D, q, q+1, \frac{q^{D+1}(q+1) - q^2 - 1}{q-1}\right), \qquad \left(D, q, q, \frac{q^2(q^D - 1)}{q-1}\right).$$

Lemma 3 (B. Fernández, R. Maleki, Š. Miklavič, G.M.)

The family of distance-regular graphs with classical parameters

$$\left(D, q, q+1, \frac{q^{D+1}(q+1)-q^2-1}{q-1}\right)$$

does not exist.

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Sketch of the proof

- $D \geq 6$: the intersection number

$$p^6_{33} = \frac{c_4 c_5 c_6}{c_1 c_2 c_3}$$

(independent of both D and β) is an integer only for $q = 2, 4$.

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(independent of both D and β) is an integer only for $q = 2, 4$.

- $D = 4, D = 5, q = 2, q = 4$: the multiplicity f_2 of the 2nd eigenvalue of Γ turns out not to be an integer.

Lemma 4 (B. Fernández, R. Maleki, Š. Miklavič, G.M.)

If $D \not\equiv 0 \pmod 6$, then the family of distance-regular graphs with classical parameters

$$\left(D, q, q, \frac{q^2(q^D - 1)}{q - 1}\right)$$

does not exist.

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Sketch of the proof

- Claim: the multiplicity f_2 of the 2nd eigenvalue of Γ is an integer only for D even.

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does not exist.

Sketch of the proof

- Claim: the **multiplicity** f_2 of the 2nd eigenvalue of Γ is an integer only for D **even**.
- Claim: the **multiplicity** f_3 of the 3rd eigenvalue of Γ is an integer only for $D \equiv 0 \pmod 6$.

Main Theorem (B. Fernández, R. Maleki, Š. Miklavič, G.M.)

Let Γ be a 1-thin, non-bipartite distance-regular graph with classical parameters $D \geq 4$, $q \geq 2$, $\alpha \neq 0$.
If Γ supports a uniform structure w.r.t. x, then it must have classical parameters

$$\left(D, q, q, \frac{q^2(q^D - 1)}{q - 1}\right), \qquad D \equiv 0 \pmod 6$$

.

Proof
It follows from previous Propositions $1, 2$ and Lemmas 1-4

Remark

The valency k_D and the multiplicity f_D of Γ (with $\alpha = q$ and $\beta = q^2(q^D - 1)/(q-1)$) respectively are

$$k_D = q^{\frac{D(D+1)}{2}+1} \prod_{i=1}^{D-1} \left(q \frac{q^D - 1}{q^i - 1} - 1 \right),$$

$$f_D = (q^D(q+1) - q) \prod_{i=2}^{D} \left(q^{i+1} \frac{q^D - 1}{q^i - 1} + 1 \right).$$

$\Rightarrow$ Computational results (Mathematica), show that they are never integers for every $q, D \leq 2000$

Conjecture

The family of distance-regular graphs with classical parameters

$$\left(D, q, q, \frac{q^2(q^D - 1)}{q - 1}\right)$$

does not exist.

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Corollary to Conjecture

Let Γ be a 1-thin, non-bipartite distance-regular graph with classical parameters $D \geq 4$, $q \geq 2$, $\alpha \neq 0$.
Then, Γ does not supports a uniform structure w.r.t. x.

Case $\alpha = 0$

Examples are *dual polar graphs* which have classical parameters

$$\left(D, q, 0, q^e\right), \;\; e \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$

$\Rightarrow$ Dual polar graphs support a uniform structure (C. Worawannotai, 2013)

Case $\alpha = 0$

Examples are *dual polar graphs* which have classical parameters

$$\left(D, q, 0, q^e\right), \;\; e \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$

$\Rightarrow$ Dual polar graphs support a uniform structure (C. Worawannotai, 2013)

Remark

- For *our* Γ (1-thin w.r.t. x), $\alpha = 0 \iff \Delta = \Delta(x)$ is a disjoint union of cliques, where $r = a_1 = \beta - 1$ and $s = -1$.
- $\alpha = 0 \Rightarrow a_i = a_1 c_i$ $(1 \leq i \leq D - 1)$

Definition

$K_{1,1,2}$: complete multipartite graph with three parts of order 1, 1, and 2, respectively

A distance-regular graph G is a *near polygon* if $a_i = a_1 c_i$ $(1 \leq i \leq D-1)$, and G does not contain $K_{1,1,2}$ as an induced subgraph.

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Theorem (A. Brouwer et al.,1989)

Let G be a distance-regular graph with classical parameters $(D, q, 0, \beta)$, $D \ge 3$. If G is a near polygon, then G is a Hamming graph $(q = 1)$ or a dual polar graph.

Theorem (B. Fernández, R. Maleki, Š. Miklavič, G.M.)

Let Γ be a non-bipartite distance-regular graph with classical parameters $D \geq 3$, $q \geq 2$, $\alpha = 0$.
Assume that Γ is 1-thin w.r.t. every vertex.
Then, Γ is a dual polar graph.

Proof

It follows from previous Remark and Theorem (A. Brouwer et al.,1989).

Thank you for your attention!