Some new extremal $\mathbb{Z}_4$-codes of lengths 32 and 40

Matteo Mravić
matteo.mravic@math.uniri.hr

Joint work with Sanja Rukavina
Faculty of Mathematics, University of Rijeka, Croatia

Binary codes

- **Binary** $[n, k]$ **code** is a k-dimensional vector subspace of a vector space $\mathbb{F}_2^n$,
- Every binary $[n, k]$ code C have **generator matrix** in form:

$$G = \left[\begin{array}{cc} F & I_k \end{array} \right],$$

  where rows of G form a basis of space C,
- Binary code C is **doubly-even** if the number of non-zero coordinates in every $v \in C$ is the multiple of 4
- The **dual code** of C is $C^\perp = \{x \in \mathbb{F}_2^n \mid \langle x, y \rangle = 0 \text{ for all } y \in C\}$,
- A code C is **self-orthogonal** if $C \subseteq C^\perp$,
- Every doubly-even binary code is self-orthogonal

$\mathbb{Z}_4$-codes

- A $\mathbb{Z}_4$-**code** C of length n is a sub-module of the $\mathbb{Z}_4$-module $\mathbb{Z}_4^n$,
- **Dual code**: $C^\perp = \left\{ x \in \mathbb{Z}_4^n \middle| \langle x, c \rangle = 0 \text{ for all } c \in C \right\}$,
- C is **self-dual** if $C = C^\perp$,
- Every self-dual $\mathbb{Z}_4$-code C can be represented with **generator matrix** of the form:

$$G = \begin{bmatrix} F & I_k + 2B \\ 2H & O \end{bmatrix},$$

  where rows form a generating set of module C,
- It is of **type** $4^k 2^{n-2k}$,
- For $x = (x_1, x_2, \ldots, x_n) \in C$, **Euclidean weight** of x is defined as

$$wt_E(x) = n_1(x) + 4n_2(x) + n_3(x),$$

  where $n_i(x) = |\{x_j | x_j = i, j \in \{1, 2, \ldots, n\}\}|$, $i = 1, 2, 3$.
- Self-dual $\mathbb{Z}_4$-code C is **Type II code** if Euclidean weight of every codeword is divisible by 8. Otherwise, it is **Type I code**.
- Type II $\mathbb{Z}_4$-code of length n exists only for $8 \big| n$

$\mathbb{Z}_4$ -codes

- The **minimum Euclidean weight** of the $\mathbb{Z}_4$ -code C is:

$$wt_E(C) = \min \left\{ wt_E(x) \middle| x \in C,\, x \neq 0 \right\}.$$

- The minimum Euclidean weight of the self-dual $\mathbb{Z}_4$ -code C is at most $8 \left\lfloor \frac{n}{24} \right\rfloor + 8$,
- If C have minimum Euclidean weight $8 \left\lfloor \frac{n}{24} \right\rfloor + 8$ then it is **extremal**,
- The **Euclidean weight enumerator** of the self-dual $\mathbb{Z}_4$ -code C is polynomial $p(x) = \sum_{i=0}^{n} A_{4i} x^{4i}$, where $A_{4i} = \left| \left\{ v \in C \middle| wt_E(v) = 4i \right\} \right|$.

Remark

There are two binary codes associated with self-dual $\mathbb{Z}_4$ -code:

- **Residue code**:

$$Res(C) = \{c \pmod 2 \mid c \in C\}.$$

- **Torsion code**:

$$Tor(C) = \{c \in \mathbb{F}_2^n | 2c \in C\}.$$

Construction of self-dual $\mathbb{Z}_4$-code from doubly-even binary code

1. Start with a doubly-even binary code C_1 ($Res(C)$):

$$G_1 = \begin{bmatrix} F & I_k \end{bmatrix},$$

2. Expand the matrix G_1 to the generator matrix of $C_1^{\perp}$ ($Tor(C)$):

$$G_2 = \begin{bmatrix} F & I_k \\ H & O \end{bmatrix},$$

3. Choose a binary $k \times k$ matrix B such that the rows of:

$$G(B) = \begin{bmatrix} F & I_k + 2B \\ 2H & O \end{bmatrix},$$

are orthogonal. The matrix $G(B)$ is the generator matrix of a self-dual $\mathbb{Z}_4$-code.

Remark

Elements above the diagonal of B can be chosen freely, and they uniquely determine elements below the diagonal. Therefore, there are $2^{\frac{k(k-1)}{2}}$ self-dual $\mathbb{Z}_4$-codes constructed from a fixed doubly-even binary $[n, k]$ code.

V. Pless, J. S. Leon, J. Fields, *All $\mathbb{Z}_4$ Codes of Type II and Length 16 Are Known*, J. Comb. Theory, Ser. A **78** (1997), 32-50.

P. Gaborit, M. Harada, *Construction of Extremal Type II Codes over $\mathbb{Z}_4$*, Des. Codes, Cryptogr. **16**(3) (1999), 257-269.

- Binary residue codes obtained from skew-symmetric Hadamard matrix of order 8

Code	[n,k,d]	0	4	8	12	16
$C_{32,1} = RM(1,5)$	$[32,6,16]$	1				62
$C_{32,2}, C_{32,7}$	$[32,9,8]$	1		28		454
$C_{32,3}$	$[32,12,4]$	1	28	84	420	3030
$C_{32,4}$	$[32,15,4]$	1	56	924	3976	22854
$C_{32,5}$	$[32,9,4]$	1	7		49	398
$C_{32,6}$	$[32,15,4]$	1	42	560	5558	20446
$C_{32,8}$	$[32,10,4]$	1	14	4	98	790
$C_{32,9}, C_{32,13}$	$[32,16,4]$	1	56	1180	11144	40774
$C_{32,10}$	$[32,7,8]$	1		4		118
$C_{32,11}$	$[32,10,8]$	1		32	112	734

Code	[n,k,d]	0	4	8	12	16
$C_{32,12}$	$[32,13,4]$	1	28	228	868	5942
$C_{32,14}$	$[32,10,8]$	1		60		902
$C_{32,15}$	$[32,10,4]$	1	8	28	56	838
$C_{32,16}$	$[32,16,4]$	1	120	1820	8008	45638
$C_{32,17}$	$[32,7,4]$	1	1		7	110
$C_{32,18}$	$[32,10,4]$	1	8	7	140	712
$C_{32,19}$	$[32,13,4]$	1	36	196	924	5878
$C_{32,20}$	$[32,10,4]$	1	1	42	63	810
$C_{32,21}$	$[32,16,4]$	1	50	1120	11438	40318

S. Ban, D. Crnković, M. Mravić and S. Rukavina, *New extremal Type II $\mathbb{Z}_4$ -codes of length 32 obtained from Hadamard matrices*, Discrete Math. Algorithms Appl. **11**(5) (2019), 1950057

Obtained extremal $\mathbb{Z}_4$-codes of length 32 with RS

- [1]There is an unique, known, extremal Type II $\mathbb{Z}_4$-code with residue code $RM(1,5)$
- In total 4 new type II codes, and 443 new Type I codes
- RS method was unable to find new codes on some binary residue codes

Type I

The binary code	The number of obtained extremal $\mathbb{Z}_4$-codes	At least inequivalent	The type	The binary residue code
$C_{32,3}$	13	10	$4^{12}2^{8}$	[32,12,4]
$C_{32,4}$	6	6	$4^{15}2^{2}$	[32,15,4]
$C_{32,8}$	35	2	$4^{10}2^{12}$	[32,10,4]
$C_{32,12}$	5	5	$4^{13}2^{6}$	[32,13,4]
$C_{32,15}$	210	2	$4^{10}2^{12}$	[32,10,4]
$C_{32,16}$	272	240	$4^{16}2^{0}$	[32,16,4]
$C_{32,18}$	44	1	$4^{10}2^{12}$	[32,10,4]
$C_{32,19}$	188	177	$4^{13}2^{6}$	[32,13,4]

Type II

The binary code	The number of obtained extremal $\mathbb{Z}_4$-codes	At least inequivalent	The type	The binary residue code
$C_{32,1}$	118	1	$4^{6}2^{20}$	[32,6,16]
$C_{32,2}$	114	1	$4^{9}2^{14}$	[32,9,8]
$C_{32,7}$	91	1	$4^{9}2^{14}$	[32,9,8]
$C_{32,10}$	296	1	$4^{7}2^{18}$	[32,7,8]
$C_{32,14}$	304	1	$4^{10}2^{12}$	[32,10,8]

[1]M. Harada, *On the Residue Codes of Extremal Type II $\mathbb{Z}_4$-Codes of Lengths 32 and 40*, Discrete Math. **(**2011), 2148-2157.

S. Ban, D. Crnković, M. Mravić and S. Rukavina, *New extremal Type II $\mathbb{Z}_4$-codes of length 32 obtained from Hadamard matrices*, Discrete Math. Algorithms Appl. **11**(5) (2019), 1950057

- Checking minimum Euclidean weight for every possible (randomly generated) matrix B (in total $2^{\frac{k(k-1)}{2}}$),
- Problem: slow computation of minimum Euclidean weight even for small lengths and dimensions of starting binary code!

n	k	One code	Search space
16	6	< 0.001s	32.768s
24	6	0.734s	6.6h,
32	6	170.830s	10.6yrs

All computations are performed by MAGMA

Neighboring codes

Definition

Let $B = [b_{s,t}]$ and $B' = [b'_{s,t}]$ be two $k \times k$ binary matrices that define generator matrices $G(B)$ and $G(B')$ of two self-dual $\mathbb{Z}_4$-codes C and C'. Let (i,j), $1 \leq i < j \leq k$, be the upper diagonal position of B such that $b_{ij} = 0$. If for $1 \leq s < t \leq k$:

$$b'_{st} = \begin{cases} b_{st}, & (s,t) \neq (i,j) \\ 1, & (s,t) = (i,j), \end{cases}$$

then we say that C' is (i,j)-neighbor of C.

Change of weight by codeword

Theorem

Let C be a self-dual $\mathbb{Z}_4$-code of length n with the generator matrix $G(B)$ as before. Let C' be the (i,j)-neighbor of C, with the generator matrix $G(B')$ as before. Let $v \in C$ be of the form:

$$v = c_i g_i + c_j g_j + \sum_{\substack{m=1 \\ m \neq i,j}}^{k} c_m g_m + \sum_{m=k+1}^{n-k} c_m g_m,$$

where g_s, $s \in \{1, 2, \dots, n-2k\}$, is the s-th row of the matrix $G(B)$. Let $v' \in C'$ be:

$$v' = c_i g_i' + c_j g_j' + \sum_{\substack{m=1 \\ m \neq i,j}}^{k} c_m g_m + \sum_{m=k+1}^{n-k} c_m g_m.$$

If c_i and c_j are both even or both odd then $wt_E(v') = wt_E(v)$.

Change of weight by codeword

Theorem

Let C, C', $G(B)$, $G(B')$, v and v' be as in the previous theorem. Let c_i and c_j be of different parity, and $s \in \{i, j\}$ such that c_s is even. Let $\sigma(c_i, c_j)$ be defined as follows:

$$\sigma(c_i, c_j) = \begin{cases} -1, & 2 \in \{c_i, c_j\}, \\ 1, & 2 \notin \{c_i, c_j\}. \end{cases}$$

Let S be the following set:

$$S = \left\{ t \in \{1, 2, \dots, k\} \setminus \{i, j\} \,\middle|\, (c_t g_t)_{n-k+s} = 2 \right\}.$$

Then, $wt_E(v') = wt_E(v) + 4r$, where r is defined as follows:

$$r = \begin{cases} (-1)^{|S|} \cdot \sigma(c_i, c_j), & b_{i,j} \neq b_{j,i} \text{ and } s = j, \\ (-1)^{|S| + b_{i,j} + b_{j,i}} \cdot \sigma(c_i, c_j), & \text{otherwise.} \end{cases}$$

$$
\begin{aligned}
v &= c_i g_i + c_j g_j + \sum_{\substack{m=1\\ m\neq i,j}}^{k} c_m g_m + \sum_{m=k+1}^{n-k} c_m g_m,\\
v' &= c_i g_i' + c_j g_j' + \sum_{\substack{m=1\\ m\neq i,j}}^{k} c_m g_m + \sum_{m=k+1}^{n-k} c_m g_m.
\end{aligned}
$$

$$
B = \begin{array}{c c}
 & \begin{array}{ccccccc} \phantom{b_{k1}} & & i & & j & & \phantom{b_{1k}} \end{array} \\
\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} &
\begin{bmatrix}
0 & \cdots & b_{i1} & \cdots & b_{1j} & \cdots & b_{1k} \\
\vdots & \ddots & \vdots & & \vdots & & \vdots \\
b_{i1} & \cdots & 0 & \cdots & \color{red}{b_{ij}} & \cdots & b_{ik} \\
\vdots & & \vdots & \ddots & \vdots & & \vdots \\
b_{j1} & \cdots & \color{red}{b_{ji}} & \cdots & 0 & \cdots & b_{jk} \\
\vdots & & \vdots & & \vdots & \ddots & \vdots \\
b_{k1} & \cdots & b_{ki} & \cdots & b_{kj} & \cdots & 0
\end{bmatrix}
\end{array}
$$

$b_{ij} = b_{ji}$				
c_i	c_j	$\lvert I\rvert \pmod 2$	$\lvert J\rvert \pmod 2$	r
0	1,3	0	$*$	1
		1	$*$	-1
1,3	0	$*$	0	1
		$*$	1	-1
2	1,3	0	$*$	-1
		1	$*$	1
1,3	2	$*$	0	-1
		$*$	1	1

$b_{ij} \neq b_{ji}$				
c_i	c_j	$\lvert I\rvert \pmod 2$	$\lvert J\rvert \pmod 2$	r
0	1,3	0	$*$	-1
		1	$*$	1
1,3	0	$*$	0	1
		$*$	1	-1
2	1,3	0	$*$	1
		1	$*$	-1
1,3	2	$*$	0	-1
		$*$	1	1

Theorem

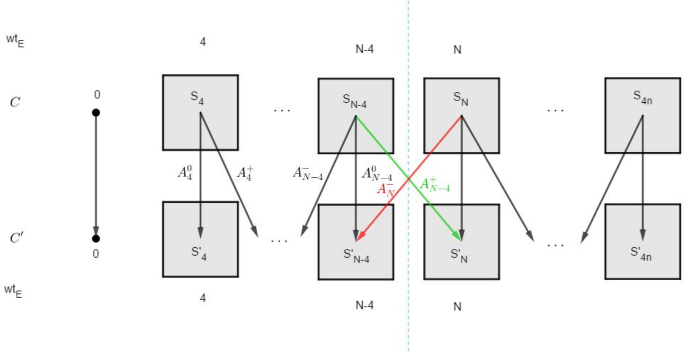
Let C be a self-dual $\mathbb{Z}_4$-code of length n, and C' its (r,s)-neighbor. Let $p(x) = \sum_{i=0}^{n} A_{4i} x^{4i}$ and $p'(x) = \sum_{i=0}^{n} A'_{4i} x^{4i}$ be the Euclidean weight enumerators of C and C', respectively. For $m \in \{0, 4, \ldots, 4n\}$, let S_m and S'_m denote the sets of codewords of Euclidean weight m from C and C', respectively. For $i = 1, 2, \ldots, n$, we define the following sets and their cardinality:

$$
\begin{aligned}
S_{4i}^{-} &= \left\{ v \in S_{4i} | v' \in S'_{4i-4} \right\}, & |S_{4i}^{-}| &= A_{4i}^{-}, \\
S_{4i}^{0} &= \left\{ v \in S_{4i} | v' \in S'_{4i} \right\}, & |S_{4i}^{0}| &= A_{4i}^{0}, \\
S_{4i}^{+} &= \left\{ v \in S_{4i} | v' \in S'_{4i+4} \right\}, & |S_{4i}^{+}| &= A_{4i}^{+},
\end{aligned}
$$

where for $v \in C$, v' is the codeword from C' with the same coefficients as v. Then the following holds:

- $A'_{4i} = A_{4i-4}^{+} + A_{4i}^{0} + A_{4i+4}^{-}$, for $i > 1$, and $A'_4 = A_4^0 + A_8^-$
- $A'_4 + A'_8 + \ldots + A'_{N-4} = A_4 + A_8 + \ldots + A_{N-4} - A_{N-4}^{+} + A_N^{-}$,

where $N \in \{8, 12, \ldots, 4n\}$.



The modification

Let N be desired minimum Euclidean weight of the self-dual $\mathbb{Z}_4$-code ($N = 8\left\lfloor \frac{n}{24} \right\rfloor + 8$ for the extremal $\mathbb{Z}_4$-codes). Then:

- Start with the arbitrary matrix B,
- In each iteration of the algorithm do the following:
  - Generate a self-dual $\mathbb{Z}_4$-code C with the generator matrix $G(B)$, and evaluate
  $$D = \left|\{v \in C | 0 < wt_E(v) < N\}\right|,$$
  - If $D = 0$ then C is extremal,
  - Determine the sets S_{N-4} and S_N ,
  - For every upper diagonal element (i, j) of the matrix B which is equal to 0 determine the (i, j)-neighbor C'. If extremality of the corresponding neighbor is undetermined, evaluate numbers A^+_{N-4}, A^-_N and $d = D - A^+_{N-4} + A^-_N$.
  - All C' that have $d = 0$ are extremal,
  - Mark all neighbors of B as checked,
  - Repeat the process with the next matrix B.

Performance of the algorithm

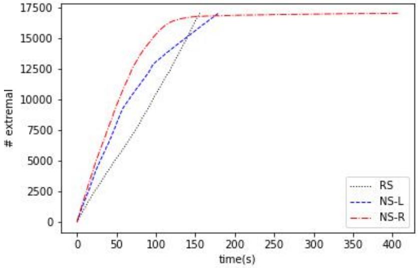


Figure: Comparison of the random search method (RS), the neighborhood search with lexicographic traversal (NS-L), and the neighborhood search with random choice of B (NS-R), on residue code $[16, 6, 4]$

- RS method: 155.844s
- NS-L better untill 127.438s,
- NS-R 95% of codes in 115.047s,
- Slowdown due to exploitation of unchecked codes.

Obtained extremal $\mathbb{Z}_4$-codes of length 32 with NS-R

The binary code	# of obtained extremal $\mathbb{Z}_4$-codes	# of nonequivalent Type I	# of nonequivalent Type II	The type	The binary residue code
$C_{32,2}$	6137	≥ 3	$\geq 1(1)$	$4^9 2^{14}$	$[32, 9, 8]$
$C_{32,3}$	161	≥ 88	$\geq 35(1)$	$4^{12} 2^8$	$[32, 12, 4]$
$C_{32,4}$	42	24	18	$4^{15} 2^2$	$[32, 15, 4]$
$C_{32,5}$	1664	≥ 2	≥ 1	$4^9 2^{14}$	$[32, 9, 4]$
$C_{32,6}$	27	0	≥ 27	$4^{15} 2^2$	$[32, 15, 4]$
$C_{32,7}$	3035	≥ 3	$\geq 1(1)$	$4^9 2^{14}$	$[32, 9, 8]$
$C_{32,8}$	532	$\geq 21(1)$	≥ 6	$4^{10} 2^{12}$	$[32, 10, 4]$
$C_{32,9}$	11	0	11	4^{16}	$[32, 16, 4]$
$C_{32,10}$	44687	≥ 3	$\geq 1(1)$	$4^7 2^{18}$	$[32, 7, 8]$
$C_{32,11}$	409	0	≥ 1	$4^{10} 2^{12}$	$[32, 10, 8]$
$C_{32,12}$	7	2	5	$4^{13} 2^6$	$[32, 13, 4]$
$C_{32,13}$	11	0	11	4^{16}	$[32, 16, 4]$
$C_{32,14}$	722	≥ 2	$\geq 1(1)$	$4^{10} 2^{12}$	$[32, 10, 8]$
$C_{32,15}$	3456	$\geq 99(1)$	≥ 24	$4^{10} 2^{12}$	$[32, 10, 4]$
$C_{32,16}$	17	8(1)	9	4^{16}	$[32, 16, 4]$
$C_{32,17}$	4800	≥ 2	$\geq 1(1)$	$4^7 2^{18}$	$[32, 7, 4]$
$C_{32,18}$	24	≥ 5	≥ 1	$4^{10} 2^{12}$	$[32, 10, 4]$
$C_{32,19}$	109	$\geq 55(2)$	$\geq 49(1)$	$4^{13} 2^6$	$[32, 13, 4]$
$C_{32,20}$	1483	≥ 7	≥ 5	$4^{10} 2^{12}$	$[32, 10, 4]$
$C_{32,21}$	8	0	8	4^{16}	$[32, 16, 4]$

- NS-R successfully constructed extremal codes from missing binary residue codes
- Non equivalence determined by invariants,
- In total 182 new Type II codes, and 762 new Type I codes

In context of known Type II codes

Type	$4^6 2^{20}$	$4^7 2^{18}$	$4^8 2^{16}$	$4^9 2^{14}$	$4^{10} 2^{12}$	$4^{11} 2^{10}$
#	1	7	27	15+1	5+37	3
References	[5]	[1, 2, 5]	[1, 5]	[1, 2, 5]	[1, 2, 5]	[5, 6]

Type	$4^{12} 2^8$	$4^{13} 2^6$	$4^{14} 2^4$	$4^{15} 2^2$	4^{16}
#	1+35	220+53	5148	356+18	134+39
References	[5]	[3, 5]	[1, 3, 5]	[3, 5]	[4, 5]

[1] S. Ban, *Construction of Extremal Type II Z4-Codes*, PhD Thesis, University of Zagreb, Zagreb, Croatia, 2019. (in Croatian)

[2] S. Ban, D. Crnković, M. Mravić, S. Rukavina, *New extremal Type II $\mathbb{Z}_4$ -codes of length 32 obtained from Hadamard matrices*, Discrete Math. Algorithms Appl. **11**(5) (2019), 1950057

[3] K. H. Chan, *Three new methods for construction of extremal Type II $\mathbb{Z}_4$ -codes*, PhD Thesis, University of Illinois at Chicago, 2012.

[4] P. Gaborit, M. Harada, *Construction of Extremal Type II Codes over $\mathbb{Z}_4$*, Des. Codes, Cryptogr. **16**(3) (1999), 257-269.

[5] M. Harada, *On the Residue Codes of Extremal Type II $\mathbb{Z}_4$ -Codes of Lengths 32 and 40*, Discrete Math. (2011), 2148-2157.

[6] V. Pless, P. Solé, Z. Qian, S. Antipolis, *Cyclic Self-Dual $\mathbb{Z}_4$ -Codes*, Finite Fields Appl. **3** (1997), 48-69.

Known Type II codes of length 40

Type	$4^7 2^{26}$	$4^8 2^{24}$	$4^9 2^{22}$	$4^{10} 2^{20}$	$4^{11} 2^{18}$	$4^{12} 2^{16}$	$4^{13} 2^{14}$
#	3	228	100	2	3	20	15
References	[1, 4]	[1, 4]	[1, 4]	[1, 4]	[1, 4]	[1, 4]	[1, 4, 5]

Type	$4^{14} 2^{12}$	$4^{15} 2^{10}$	$4^{16} 2^8$	$4^{17} 2^6$	$4^{18} 2^4$	$4^{19} 2^2$	4^{20}
#	5	3	1	134	902	432	94343
References	[1, 4]	[1, 4]	[4]	[2, 4]	[1, 2, 4]	[2, 4]	[3, 5]

[1] S. Ban, S. Rukavina, *On some new extremal Type II $\mathbb{Z}_4$ -codes of length 40*, Math. Commun. **25**(2) (2020), 253-268.

[2] K. H. Chan, *Three new methods for construction of extremal Type II $\mathbb{Z}_4$ -codes*, PhD Thesis, University of Illinois at Chicago, 2012.

[3] M. Harada, *Extremal Type II $\mathbb{Z}_4$-codes constructed from binary doubly even self-dual codes of length* 40, Discrete Math. **340**(10) (2017), 253-268.

[4] M. Harada, *On the Residue Codes of Extremal Type II $\mathbb{Z}_4$ -Codes of Lengths 32 and 40*, Discrete Math. **(**2011), 2148-2157.

[5] V. Pless, P. Solé, Z. Qian, S. Antipolis, *Cyclic Self-Dual $\mathbb{Z}_4$ -Codes*, Finite Fields Appl. **3** (1997), 48-69.

Binary residue codes of length 40

Code	[n,k,d]	0	4	8	12	16	20
$C_{40,1}$	$[40,7,12]$	1			1	11	102
$C_{40,2}$	$[40,7,16]$	1				15	96
$C_{40,3}$	$[40,10,4]$	1	6	1	10	150	688
$C_{40,4}$	$[40,10,12]$	1			18	223	540
$C_{40,5}$	$[40,11,4]$	1	10	6	22	313	1344
$C_{40,6}$	$[40,11,12]$	1			34	479	1020
$C_{40,7}$	$[40,11,12]$	1			42	447	1068
$C_{40,8}$	$[40,15,4]$	1	37	175	688	5296	20374
$C_{40,9}$	$[40,15,8]$	1		10	634	7589	16300
$C_{40,10}$	$[40,15,8]$	1		6	658	7529	16380
$C_{40,11}$	$[40,16,4]$	1	47	313	1548	10694	40330
$C_{40,12}, C_{40,13}$	$[40,20,8]$	1	0	285	21280	239970	525504
$C_{40,14}$	$[40,20,4]$	1	190	4845	38760	125970	709044

[1] S. Ban, S. Rukavina, *On some new extremal Type II $\mathbb{Z}_4$ -codes of length 40*, Math. Commun. **25**(2) (2020), 253-268.

[2] M. Harada, *On the Residue Codes of Extremal Type II $\mathbb{Z}_4$ -Codes of Lengths 32 and 40*, Discrete Math. **(**2011), 2148-2157.

[3] D. Crnković, M. Mravić, S. Rukavina, *Construction of extremal $\mathbb{Z}_4$ -codes using a neighborhood search algorithm* (submitted)

Obtained extremal $\mathbb{Z}_4$-codes of length 40

The binary code	The type	# of obtained extremal $\mathbb{Z}_4$ codes	# of nonequivalent Type I	# of nonequivalent Type II
$C_{40,1}$	$4^7 2^{26}$	488	≥ 4	≥ 1
$C_{40,2}$	$4^7 2^{26}$	959	≥ 3	0
$C_{40,3}$	$4^{10} 2^{20}$	17	≥ 6	0
$C_{40,4}$	$4^{10} 2^{20}$	129	≥ 3	≥ 3
$C_{40,5}$	$4^{11} 2^{18}$	7	7	0
$C_{40,6}$	$4^{11} 2^{18}$	31	0	$\geq 5(1)$
$C_{40,7}$	$4^{11} 2^{18}$	13	0	≥ 4
$C_{40,8}$	$4^{15} 2^{10}$	37	≥ 17	0
$C_{40,9}$	$4^{15} 2^{10}$	7	0	7
$C_{40,10}$	$4^{15} 2^{10}$	9	0	9
$C_{40,11}$	$4^{16} 2^{8}$	104	≥ 14	≥ 1
$C_{40,12}$	4^{20}	7	0	7
$C_{40,13}$	4^{20}	1	0	1
$C_{40,14}$	4^{20}	4194	≥ 4090	≥ 3

- In total 4144 Type I codes and 40 Type II codes,
- To the best of our knowledge, Type I codes of types $4^7 2^{26}, 4^{11} 2^{18}, 4^{15} 2^{10}, 4^{16} 2^8$ have not been constructed before.

Known Type II codes of length 40

Type	$4^7 2^{26}$	$4^8 2^{24}$	$4^9 2^{22}$	$4^{10} 2^{20}$	$4^{11} 2^{18}$	$4^{12} 2^{16}$	$4^{13} 2^{14}$
#	3+1	228	100	2+3	3+8	20	15
References	[1, 4]	[1, 4]	[1, 4]	[1, 4]	[1, 4]	[1, 4]	[1, 4, 5]

Type	$4^{14} 2^{12}$	$4^{15} 2^{10}$	$4^{16} 2^{8}$	$4^{17} 2^{6}$	$4^{18} 2^{4}$	$4^{19} 2^{2}$	4^{20}
#	5	3+16	1+1	134	902	432	94343+11
References	[1, 4]	[1, 4]	[4]	[2, 4]	[1, 2, 4]	[2, 4]	[3, 5]

[1] S. Ban, S. Rukavina, *On some new extremal Type II $\mathbb{Z}_4$ -codes of length 40*, Math. Commun. **25**(2) (2020), 253-268.

[2] K. H. Chan, *Three new methods for construction of extremal Type II $\mathbb{Z}_4$ -codes*, PhD Thesis, University of Illinois at Chicago, 2012.

[3] M. Harada, *Extremal Type II $\mathbb{Z}_4$-codes constructed from binary doubly even self-dual codes of length* 40, Discrete Math. **340**(10) (2017), 253-268.

[4] M. Harada, *On the Residue Codes of Extremal Type II $\mathbb{Z}_4$ -Codes of Lengths 32 and 40*, Discrete Math. **(**2011), 2148-2157.

[5] V. Pless, P. Solé, Z. Qian, S. Antipolis, *Cyclic Self-Dual $\mathbb{Z}_4$ -Codes*, Finite Fields Appl. **3** (1997), 48-69.

Thank you!