On the additivity of 2-(v, k, λ) designs

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Definition

A 2-(v, k, λ) *design* is a pair $(V, \mathcal{B})$ such that

- V is a set of v points;
- $\mathcal{B}$ is a collection of k-subsets of V (called blocks);
- each 2-subset of V is contained in λ blocks.

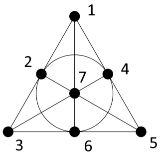


Figure: The Fano plane. 2-$(7, 3, 1)$ design.

- A 2-design is *symmetric* if $|V| = |\mathcal{B}|$.
- A *Steiner system* is a design with $\lambda = 1$.

Definition (Caggegi, Falcone, Pavone, 2017)

A design $(V, \mathcal{B})$ is *additive* under an abelian group G if

- $V \subseteq G$ and
- $\sum_{x \in B} x = 0, \quad \forall B \in \mathcal{B}.$

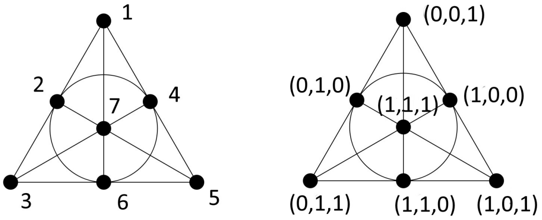


Figure: The Fano plane is additive under $\mathbb{Z}_2^3$.

- check for one one line: $(0,0,1)+(0,1,0)+(0,1,1)=(0,0,0)$

Definition (Caggegi, Falcone, Pavone, 2017)

A design $(V, \mathcal{B})$ is *additive* under an abelian group G if

- $V \subseteq G$ and
- $\sum_{x \in B} x = 0, \quad \forall B \in \mathcal{B}.$

Examples:

Parameters	Group	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$	AG$_1(n, p^m)$, points-lines design of AG(n, p^m)
$([n+1]_2, 3, 1)$	$\mathbb{Z}_2^n$	PG$_1(n, 2)$, points-lines design of PG$(n, 2)$

The number of points of PG(n, q) is denoted by $[n+1]_q = \frac{q^{n+1}-1}{q-1}$

Definition (Cameron, 1974. Delsarte, 1976.)

A 2-(v, k, λ) *design over* $\mathbb{F}_q$ is a pair $(V, \mathcal{B})$ such that

- V is the set of points of $\mathrm{PG}(v-1, q)$
- $\mathcal{B}$ is a collection of $(k-1)$-dimensional subspaces $\mathrm{PG}(v-1, q)$ (blocks)
- each line is contained in λ blocks.

Properties:

- The Fano plane is $(3, 2, 1)$ design over $\mathbb{F}_2$
- (v, k, λ) design over $\mathbb{F}_q$ is a classical $([v]_q, [k]_q, \lambda)$ design
- (v, k, λ) design over $\mathbb{F}_2$ is additive under $\mathbb{Z}_2^v$

Parameters	Description	Reference
$([v]_2, 7, 7)$	$(v, 3, 7)$ design over $\mathbb{F}_2$ for all v odd	Thomas, 1987 + Buratti, A.N., 2019
$(8191, 7, 1)$	$(13, 3, 1)$ design over $\mathbb{F}_2$	Braun, Etzion, Ostergaard, Vardy, Wassermann, 2017

Additivity of...

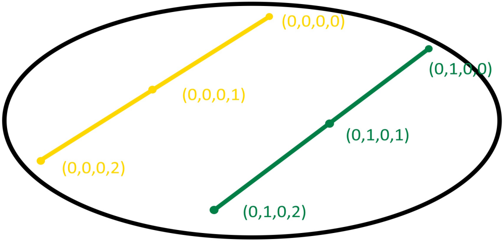
- k-parallel designs
- $\mathrm{PG}_d(n, q)$
- (cyclic) symmetric designs
- Steiner 2-designs

- A.N. The first example of a simple 2-(81, 6, 2) design. Examples and Counterexamples, 1 (2021)
- A.N., M. Buratti, Super-regular Steiner 2-designs. Finite Fields and Their Applications Volume 85, 102116 (2023)
- A.N., M. Buratti, Additivity of symmetric and subspace designs, arXiv:2307.08134

Additivity of k-parallel designs

Simple $(81, 6, 2)$ design

- construct a simple additive $(81, 6, 2)$ design
- the only known $(81, 6, 2)$ design has repeated blocks (Hanani, 1975)
- we note: every union of two parallel lines of AG$(4, 3)$ is a zero-sum 6-subset of $\mathbb{Z}_3^4$



Simple $(81, 6, 2)$ design

- 432 blocks are obtained from 16 orbits of $\mathbb{Z}_3^4$ of size 27 (representatives bellow)
- it is additive!
- it is simple
- the only known $(81, 6, 2)$ design has repeated blocks (Hanani, 1975)

$$\{(0,0,0,0),(0,0,0,1),(0,0,0,2),(0,1,0,0),(0,1,0,1),(0,1,0,2)\}$$
$$\{(0,0,0,0),(0,0,1,1),(0,0,2,2),(2,1,0,0),(2,1,1,1),(2,1,2,2)\}$$
$$\{(0,0,0,0),(0,1,1,1),(0,2,2,2),(0,0,1,0),(0,1,2,1),(0,2,0,2)\}$$
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$$\{(0,0,0,0),(1,2,2,0),(2,1,1,0),(0,2,2,1),(1,1,1,1),(2,0,0,1)\}$$

Simple $(81, 6, 2)$ design

- 432 blocks are obtained from 16 orbits of $\mathbb{Z}_3^4$ of size 27 (representatives bellow)
- it is additive!
- it is simple
- the only known $(81, 6, 2)$ design has repeated blocks (Hanani, 1975)

[A.N., Examples and Counterexamples, 2021]

Parameters	Group	Description
$(81, 6, 2)$	$\mathbb{Z}_3^4$	each block is a union of two parallel lines of AG$(4, 3)$

Definition (k-parallel design)

A (q^n, kq, λ) design $(V, \mathcal{B})$ is k-parallel if

- V is the set of points of AG(n, q),
- each block $B \in \mathcal{B}$ is union of k parallel lines of AG(n, q).

Every k-parallel design is additive under $\mathbb{F}_{q^n}$.

Theorem

Let G be a t-transitive group acting on the set X, $|X| = v$, and let B be a subset X, $|B| = k$.
Then the pair $(X, \mathcal{B})$, where $\mathcal{B}$ is the orbit of B under G

$$\mathcal{B} = \{\alpha(B) \quad | \quad \alpha \in G\}$$

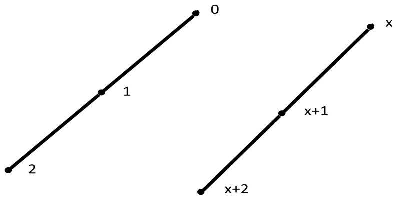
is a t-(v, k, λ) designs for $\lambda = \frac{|G|}{|Stab_G(B)|} \cdot \frac{k(k-1)}{v(v-1)}$.

Natural choice of G to obtain k-parallel designs if the group G of affinities of $\mathbb{F}_q$ which acts sharply 2-transitively on $\mathbb{F}_q$

$$G = \{\alpha_{mt} : \mathbb{F}_q \to \mathbb{F}_q, \alpha_{mt}(x) = mx + t \quad | \quad m \in \mathbb{F}_q \setminus \{0\}, t \in \mathbb{F}_q\}$$

- The group G of affinities of $\mathbb{F}_{27}$ is sharply 2-transitive on $\mathbb{F}_{27}$
- The following 6-subset B of $\mathbb{F}_{27}$ is a union of 2 parallel lines of AG$(3, 3)$

$$B = \{0, 1, 2, x, x + 1, x + 2\}, \qquad x \in \mathbb{F}_{27} \setminus \{0, 1, 2\}$$



- The stablizer of B in G has order 6

$$Stab_G(B) = \langle \alpha_{11}, \alpha_{-1,x} \rangle$$

- Compute the parameter λ:

$$\lambda = \frac{|G|}{|Stab_G(B)|} \cdot \frac{k(k-1)}{v(v-1)} = \frac{27 \cdot 26}{6} \cdot \frac{6 \cdot 5}{27 \cdot 26} = 5$$

- Choices for x: $\mathbb{F}_{27} \setminus \{0, 1, 2\}$

- We obtain two non isomorphic 2-parallel $(27, 6, 5)$ designs

	$\mathcal{D}_1$	$\mathcal{D}_2$	Abel	Hanani
$\lvert Aut(\mathcal{D})\rvert$	2106	702	78	78
$\lvert B_i \cap B_j\rvert = 0$	1404	1404	1040	702
$\lvert B_i \cap B_j\rvert = 1$	2106	2106	3198	3900
$\lvert B_i \cap B_j\rvert = 2$	3159	3159	2067	1911
$\lvert B_i \cap B_j\rvert = 3$	117	117	481	117
$\lvert B_i \cap B_j\rvert = 4$	0	0	0	78
$\lvert B_i \cap B_j\rvert = 5$	0	0	0	78

Parameters	Group	Description
$(81, 6, 2)$	$\mathbb{F}_{3^4}$	each block is a union of two parallel lines of AG$(4, 3)$
$(27, 6, 5)$	$\mathbb{F}_{3^3}$	each block is a union of two parallel lines of AG$(3, 3)$

Definition (Difference set and difference family)

Let G be an additive group.
A k-subset D of G is a (G, k, λ) *difference set* if the list of differences of D covers each non-zero element of G λ times:

$$\Delta D = \{x - y \,:\, x \neq y, x, y \in D\} = \lambda\,(G \setminus \{0\})$$

A collection of k-subsets $\mathcal{F} = \{D_1, \ldots, D_t\}$ of G is a (G, k, λ) *difference family* if the

list of differences of the blocks covers each non-zero element of G λ times:

$$\Delta\mathcal{F} = \uplus \Delta D_i = \lambda\,(G \setminus \{0\})$$

- Let $G = \mathbb{Z}_7$ and let $D = \{0, 1, 3\} \subset G$
- Difference table of D:

	0	1	3
0	$\bullet$	**6**	**4**
1	**1**	$\bullet$	**5**
3	**3**	**2**	$\bullet$

- $\Rightarrow \{D\}$ is a $(\mathbb{Z}_7, 3, 1)$-DF.
- The translates of D form the block-set of a G-regular $(7, 3, 1)$ design (Fano plane)

$$\{0,1,3\},\ \{1,2,4\},\ \{2,3,5\},\ \{3,4,6\},\ \{4,5,0\},\ \{5,6,1\},\ \{6,0,2\}$$

Theorem (Buratti, A.N., 202?)

If there exists a (q, k, λ) *difference family in* $\mathbb{F}_q$ *then there exists a* k*-parallel* (q^n, kq, μ) *design with*

$$\mu = \frac{\lambda(kq-1)}{k-1},$$

for every $n \geq 2$.

An example.

- consider the difference set $D = \{0, 1, 3\}$ with parameters $(7, 3, 1)$
- applying the theorem with $n = 2$, we obtain a 3-parallel $(7^2, 7 \cdot 3, 10)$ design
- not isomorphic to the design of Abel, 1996

Parameters	Group	Description
$(81, 6, 2)$	$\mathbb{F}_{3^4}$	each block is a union of two parallel lines of AG$(4, 3)$
$(27, 6, 5)$	$\mathbb{F}_{3^3}$	each block is a union of two parallel lines of AG$(3, 3)$
$(49, 21, 10)$	$\mathbb{F}_{7^2}$	each block is a union of three parallel lines of AG$(2, 7)$
(q^n, kq, μ)	$\mathbb{F}_{q^n}$	k-parallel designs from difference families

Corollary [Buratti, A.N., 202?]

Parameters	Description	Reference
$(q^n, 2q, 2q-1)$	$(q, 2, 1)$ DF, q odd	patterned starter
$(q^n, 3q, \frac{3q-1}{2})$	$(q, 3, 1)$ DF, $q \equiv 1 \pmod 6$	Peltesohn, 1938
$(q^n, 4q, \frac{4q-1}{3})$	$(q, 4, 1)$ DF, $q \equiv 1 \pmod{12}$	Chen, Zhu, 1999
$(q^n, 5q, \frac{5q-1}{4})$	$(q, 5, 1)$ DF, $q \equiv 1 \pmod{20}$	Chen, Zhu, 1999
$(q^n, 6q, \frac{6q-1}{5})$	$(q, 6, 1)$ DF, $q \equiv 1 \pmod{30}$ except possibly $q = 61$	Chen, Zhu, 1998
$(q^n, \frac{q(q-1)}{2}, \frac{q^2-q-2}{2})$	$(q, \frac{q-1}{2}, \frac{q-3}{4})$ DS, $q \equiv 3 \pmod 4$	Paley difference set
$(q^n, kq, kq-1)$	$(q, k, k-1)$ DF, $q \equiv 1 \pmod k$	Wilson, 1972
$(q^n, kq, \frac{kq-1}{2})$	$(q, k, \frac{k-1}{2})$ DF, $q \equiv 1 \pmod k$, q, k odd	Wilson, 1972
$(q^n, kq, \frac{k(kq-1)}{k-1})$	(q, k, k) DF, $q \equiv 1 \pmod{k-1}$	Wilson, 1972
$(q^n, kq, \frac{k(kq-1)}{2(k-1)})$	$(q, k, \frac{k}{2})$ DF, $q \equiv 1 \pmod{k-1}$	Wilson, 1972

The group is always $\mathbb{F}_{q^n}$.

Additivity of symmetric designs and $\mathsf{PG}_d(n, q)$

Definition

$(V, \mathcal{B})$ is additive under an abelian group G if $V \subseteq G$ and $\sum_{x \in B} x = 0, \forall B \in \mathcal{B}$.

- strongly additive under G if $\mathcal{B} = \{B \in \binom{G}{k} \mid \sum_{x \in B} x = 0\}$
- strictly additive under G if $V = G$
- almost strictly additive under G if $V = G \setminus \{0\}$

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$		✓		$\mathrm{AG}_1(n, p^m)$
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	$\mathrm{PG}_1(n-1, 2)$
$(2^v - 1, 2^k - 1, \lambda)$	$\mathbb{Z}_2^v$			✓	(v, k, λ) design over $\mathbb{F}_2$
(q^n, kq, μ)	$\mathbb{F}_{q^n}$		✓		k-parallel

Theorem (Buratti, A.N., 202?)

- *Every design* $PG_d(n,q)$ *is additive under* $\mathbb{F}_q^{n+1}$.
- *Every design* $PG_d(n,q)$ *is strongly additive under* $\mathbb{Z}_{q^d}^{[n+1]_q}$.

Theorem (Buratti, A.N., 202?)

- *A symmetric* (v,k,λ) *design is strongly additive under* $\mathbb{Z}_{k-\lambda}^{v}$.
- *Let* $\mathcal{D}$ *be a cyclic symmetric* (v,k,λ) *design and let* p *be a prime dividing* $k-\lambda$ *but not* v. *Then* $\mathcal{D}$ *is additive under* $\mathbb{Z}_p^t$ *with* $t = ord_v(p)$.

[Caggegi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Al. str.	Description
$(2^n-1,3,1)$	$\mathbb{Z}_2^n$	✓		✓	$\mathrm{PG}_1(n-1,2)$
$([2]_q,q+1,1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			$\mathrm{PG}_1(2,q)$
(v,k,λ)	G	✓			symmetric design
(v,k,λ)	$\mathbb{Z}_k \times \mathbb{Z}_{k-\lambda}^{\frac{v-1}{2}}$	✓			symmetric design, $k-\lambda \nmid k$, prime

[Buratti, A.N, 202+]

Parameters	Group	Strongly	Strictly	Al. str.	Description
$([n+1]_q,[d+1]_q,\lambda)$	$\mathbb{F}_q^{n+1}$				$\mathrm{PG}_d(n,q)$
$([n+1]_q,[d+1]_q,\lambda)$	$\mathbb{Z}_{q^d}^{[n+1]_q}$	✓			$\mathrm{PG}_d(n,q)$
(v,k,λ)	$\mathbb{Z}_{k-\lambda}^v$	✓			symmetric design
(v,k,λ)	$\mathbb{Z}_p^t$				cyclic symmetric design, p a prime dividing $k-\lambda$ but not v, $t=ord_v(p)$.

Definition (Caggagi, Falcone, Pavone, 2017)

A design $(V, \mathcal{B})$ is *additive* under an abelian group G if there exists an injective map

$$f : V \to G$$

such that $f(B)$ is zero-sum for every block $B \in \mathcal{B}$.

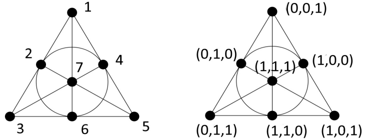


Figure: $f : V \to \mathbb{Z}_2^3$

Every cyclic symmetric (v, k, λ) design is of the form $(\mathbb{Z}_v, \{D + i \mid 0 \leq i \leq v - 1\})$ where D is a cyclic (v, k, λ) difference set.

An incidences structure $(V, \mathcal{B})$ is *cyclic* if there exists a cyclic permutation on V leaving $\mathcal{B}$ invariant.

Theorem (Buratti, A.N., 202?)

Let $\mathcal{D}$ be a cyclic symmetric (v, k, λ) design and let p be a prime dividing $k - \lambda$ but not v. Then $\mathcal{D}$ is additive under $\mathbb{Z}_p^t$ with $t = ord_v(p)$.

Proof:

- Let g be a generator of the subgroup of $\mathbb{F}_{p^t}^*$ of order v and consider the injective maps f_1 and f_{-1} defined as follows:

$$f_1 : x \in \mathbb{Z}_v \longrightarrow g^x \in \mathbb{F}_{p^t}, \qquad f_{-1} : x \in \mathbb{Z}_v \longrightarrow g^{-x} \in \mathbb{F}_{p^t}.$$

- Consider the two sums

$$\sigma_1 := \sum_{d \in D} f_1(d) = \sum_{d \in D} g^d, \qquad \sigma_{-1} := \sum_{d \in D} f_{-1}(d) = \sum_{d \in D} g^{-d}$$

- Calculate their product $\sigma_1 \cdot \sigma_{-1} = (k - \lambda) + \lambda \frac{g^v - 1}{g - 1} = 0$
- Therefore

$$\sigma_1 = 0, \quad \text{or} \quad \sigma_{-1} = 0$$

- Since

$$\sum_{b \in B} f_1(b) = \sum_{d \in D} g^{d+i} = \sigma_1 \cdot g^i \quad \text{and} \quad \sum_{b \in B} f_{-1}(b) = \sum_{d \in D} g^{-(d+i)} = \sigma_{-1} \cdot g^{-i}$$

- Either f_1 or f_{-1} is the map we are looking for □

Example

PG$(2,3)$, the projective plane of order 3, is additive under $\mathbb{Z}_3^3$.

- Singer $(13,4,1)$ difference set $D = \{0,1,3,9\}$
- $(\mathbb{Z}_{13}, \mathcal{B})$ is cyclic symmetric design with parameters $(13,4,1)$

$$\{D + i \mid 0 \leq i \leq 12\}$$

- Let r be a root of the primitive polynomial $x^3 + 2x^2 + 1$ over $\mathbb{F}_3$
- Taking r as primitive element of $\mathbb{F}_{3^3}$, a generator of the subgroup of $\mathbb{F}_{3^3}^*$ of order 13 is $g = r^2$
- We check

$$\sigma_1 = \sum_{d\in D} f_1(d) = \sum_{d\in D} g^d = g^0 + g^1 + g^3 + g^9 = r^0 + r^2 + r^6 + r^{18} =$$

$$= (0,0,1) + (1,0,0) + (2,2,0) + (0,1,1) = (0,0,2)$$

- and

$$\sigma_{-1} = \sum_{d\in D} f_{-1}(d) = \sum_{d\in D} g^{-d} = g^0 + g^{-1} + g^{-3} + g^{-9} = r^0 + r^{-2} + r^{-6} + r^{-18} =$$

$$= (0,0,1) + (0,2,1) + (2,0,2) + (1,1,2) = (0,0,0)$$

- $f_{-1} : x \in \mathbb{Z}_{13} \longrightarrow g^{-x} \in \mathbb{F}_{3^3}$

The point-hyperplane design of $\mathrm{PG}(2,3)$ is additive under $\mathbb{Z}_3^3$.

- In other words, $\mathrm{PG}(2,3)$ can be seen as the design $(V, \mathcal{B})$ where

$$V = \{001, 100, 122, 220, 112, 121, 120, 020, 201, 011, 202, 111, 021\}$$

- and where $\mathcal{B}$ consists of the following zero-sum blocks

$$\begin{array}{ccc} \{001, 021, 202, 112\}, & \{021, 111, 011, 220\}, & \{111, 202, 201, 122\}, \\ \{202, 011, 020, 100\}, & \{011, 201, 120, 001\}, & \{201, 020, 121, 021\}, \\ \{020, 120, 112, 111\}, & \{120, 121, 220, 202\}, & \{121, 112, 122, 011\}, \\ \{112, 220, 100, 201\}, & \{220, 122, 001, 020\}, & \{122, 100, 021, 120\}, \\ & \{100, 001, 111, 121\} & \end{array}$$

- There is a $(143, 71, 35)$ difference set $\Rightarrow$ cyclic symmetric $(143, 71, 35)$ design
- The prime divisor of the order $k - \lambda = 71 - 35 = 36 = 2^2 \cdot 3^2$ are 2 and 3
- $ord_{143}(2) = 60$
- $ord_{143}(3) = 15$

Example

The cyclic symmetric $(143, 71, 35)$ design is additive under $\mathbb{Z}_2^{60}$ and under $\mathbb{Z}_3^{15}$ at the same time.

[Caggegi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
(v, k, λ)	G	✓			symmetric design
(v, k, λ)	$\mathbb{Z}_k \times \mathbb{Z}_{k-\lambda}^{\frac{v-1}{2}}$	✓			symmetric design, $k - \lambda \not| k$, prime

Theorem (Buratti, A.N., 202?)

Every design $PG_d(n,q)$ *is* *strongly* *additive under* $\mathbb{Z}_{q^d}^{[n+1]_q}$.

Proof:

- set $v = [n+1]_q$ and $k = [d+1]_q$
- let $\mathcal{P} = \{x_1, \dots, x_v\}$ be an ordering of the points of PG(n,q)
- let $\mathcal{H} = \{\pi_1, \dots, \pi_v\}$ be an ordering of its hyperplanes
- consider the $v \times v$ matrix $M = (m_{i,j})$ with entries in $\mathbb{Z}_{q^d}$ defined by

$$m_{i,j} = \begin{cases} 0 & \text{if } x_i \in \pi_j \\ 1 & \text{if } x_i \notin \pi_j \end{cases}$$

- let M_i denote the i-th row of M
- consider the injective map

$$f : x_i \in \mathcal{P} \longrightarrow M_i \in \mathbb{Z}_{q^d}^v$$

- to prove the assertion, we prove that the following equivalence holds

$$S \text{ is a } d\text{-subspace of PG}(n,q) \iff S \in \binom{\mathcal{P}}{k} \text{ and } f(S) \text{ is zero-sum.}$$

- $\square$

Additivity of Steiner 2-designs

Known inifinite families of additive Steiner designs [Caggegi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$		✓		$\mathrm{AG}_1(n, p^m)$
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	$\mathrm{PG}_1(n-1, 2)$
$([2]_q, q+1, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			$\mathrm{PG}_1(2, q)$

All these examples have $k = p^m$ or $k = p^m + 1$

Definition

$(V, \mathcal{B})$ is additive under an abelian group G if $V \subseteq G$ and $\sum_{x \in B} x = 0, \forall B \in \mathcal{B}$.

- strictly additive under G if $V = G$

Proposition

A strictly additive $(v, k, 1)$ design with $v \equiv 2 \pmod 4$ does not exist.

Main ingredient of the proof:

- Group G must be zero-sum!
- if G is an abelian group of order v, then

$$\sum_{g \in G} g = \begin{cases} \text{the involution of } G & \text{if } G \text{ is binary (has only one involution)} \\ 0 & \text{otherwise} \end{cases}$$

- a group G of order $v \equiv 2 \pmod 4$ is necessarily binary

Proposition

A strictly additive $(v, k, 1)$ design with $v \equiv 2 \pmod 4$ does not exist.

Corollary

Strictly additive $(v, 6, 1)$ designs with values $v = 60n + 6$ and $v = 60n + 46$ do not exist.

One more necessary condition...

Proposition

If a strictly additive $(v, k, 1)$ design exists, then every prime factor of v must be a divisor of k.

Smallest open problems

- Unknown existence of a $(81, 6, 1)$ strictly additive design
- Unknown existence of a $(256, 6, 1)$ strictly additive design

First try:

Let $\mathcal{F} = \{D_1, \ldots, D_t\}$ be a $(v, k, 1)$-DF in G.

The set of all the translates of the base blocks of $\mathcal{F}$ form the block-set of a G-regular $(v, k, 1)$ design

$$\mathcal{B} = \{B_i = D_i + g \,:\, 1 \leq i \leq t, \quad g \in G\}$$

A DF in G is *additive* if all its members are zero-sum.

- Possible idea: Choose blocks $D_1, \ldots, D_t$ such that $\sum_{x \in D_i} x = 0$
- But for $B_i = D_i + g$ we have

$$\sum_{x \in B_i} x = \left(\sum_{x \in D_i} x \right) + kg = kg \qquad \forall g \in G$$

- Hence

$$\sum_{x \in B_i} x \neq 0 \quad \text{unless} \quad kg = 0$$

- This is why we need that $o(g) \mid k \quad \forall g \in G$

$\Rightarrow\Leftarrow$ v and k are coprime

Definition

Let G be a group of order v, and let H be a subgroup of G of order h.
Let $\mathcal{F} = \{D_1, \ldots, D_t\}$ be a set of k-sets on G.

- $\mathcal{F}$ is a $(v, k, 1)$ *(ordinary) difference family* in G if $\Delta\mathcal{F} = G \setminus \{0\}$.
- $\mathcal{F}$ is a $(v, k, 1)$ *strong difference family* in G if $\Delta\mathcal{F} = G$.
- $\mathcal{F}$ is a $(v, k, h, 1)$ **difference family in** G **relative** to H if $\Delta\mathcal{F} = G \setminus H$.

Lemma

Let G be a zero-sum group of order k and let $q \equiv 1 \pmod{k-1}$ be a power of a prime divisor p of k.

If there exists an additive $(G \times \mathbb{F}_q, G \times \{0\}, k, 1)$-DF, then there exists a **strictly additive** $(kq^n, k, 1)$ **design under** $G \times \mathbb{F}_{q^n}$ *for every $n \geq 1$.*

Blocks of the constructed G-regular design are are all the translates of the base blocks of $\mathcal{F}$ together with all the right cosets of all the members of $G \times \{0\}$

$$\mathcal{B} = \{D_i + g\} \bigcup \{G \times \{y\}\}$$

- construct strictly additive $(125, 5, 1)$ under $\mathbb{Z}_5 \times \mathbb{F}_{25}$
- design can be realized by means of an additive

$$(\mathbb{Z}_5 \times \mathbb{F}_{25}, \mathbb{Z}_5 \times \{0\}, 5, 1)\text{-DF}.$$

- the base the blocks of $\mathcal{F}$ written in additive notation of $\mathbb{Z}_5^3$, are the following:

$$\begin{aligned}
D_1 &= \{(0,0,0),(1,0,1),(1,0,4),(4,1,0),(4,4,0)\}\\
D_2 &= \{(0,0,0),(1,4,3),(1,1,2),(4,4,2),(4,1,3)\}\\
D_3 &= \{(0,0,0),(1,3,2),(1,2,3),(4,4,4),(4,1,1)\}\\
D_4 &= \{(0,0,0),(1,0,2),(1,0,3),(4,2,0),(4,3,0)\}\\
D_5 &= \{(0,0,0),(1,3,1),(1,2,4),(4,3,4),(4,2,1)\}\\
D_6 &= \{(0,0,0),(1,1,4),(1,4,1),(4,3,3),(4,2,2)\}
\end{aligned}$$

- $\mathcal{F}$ gives rise to a strictly additive $(125, 5, 1)$ design under $\mathbb{Z}_5 \times \mathbb{F}_{25}$
- Note: this design is not the design $\mathrm{AG}_1(3, 5)$

Definition

A Steiner 2-design is *G-super-regular* if it is

- is strictly additive under an abelian group G (the point set is exactly G) and
- G-regular (any translate of any block is a block as well)

New examples [Buratti, A.N., 2023]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(5^3, 5, 1)$	$\mathbb{F}_{5^3}$		✓		not isomorphic to $\mathrm{AG}_1(3,5)$
$(7^3, 7, 1)$	$\mathbb{F}_{7^3}$		✓		not isomorphic to $\mathrm{AG}_1(3,7)$
$(p^n, p, 1)$	$\mathbb{F}_{p^n}$		✓		$p \in \{5,7\}$, $n \geq 3$, not isomorphic to $\mathrm{AG}_1(n,p)$

[Caggegi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$		✓		$\mathrm{AG}_1(n, p^m)$
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	$\mathrm{PG}_1(n-1, 2)$
$([2]_q, q+1, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			$\mathrm{PG}_1(2, q)$

Theorem (Buratti, A.N., 2023)

Let $k \geq 3$, $k \not\equiv 2 \pmod 4$ and $k \neq 2^n \cdot 3 \geq 12$.
There are infinitely many values of v for which there exists a super-regular $(v, k, 1)$ design.

- the group is $G \times \mathbb{F}_q$, where G is a zero-sum group of order k and q a power of a prime divisor of k
- $(kq^n, k, 1)$ design for every $n \geq 1$

Few ideas from the proof (1).

- $[k \not\equiv 2 \pmod 4]$ G abelian group od order k such that $\sum_{x \in G} = 0$
- If you can construct $(kp^n, k, k, 1)$-DF in $G \times \mathbb{F}_{p^n}$ relative to $G \times \{0\}$, p a prime divisor of k:

$$\Delta D_1 \cup \cdots \cup \Delta D_t = G \times \mathbb{F}_{q^n} \setminus G \times \{0\}$$

- such that

$$\sum_{x \in D_i} x = 0$$

- then we get a Steiner design with $\mathcal{B} = \{D_i + g\} \bigcup \{G \times \{y\}\}$

Theorem (Buratti, A.N., 2023)

Let $k \geq 3$, $k \not\equiv 2 \pmod 4$ and $k \neq 2^n \cdot 3 \geq 12$.
There are infinitely many values of v for which there exists a super-regular $(v, k, 1)$ design.

Few ideas from the proof (2).

- Does such DF exists?
- $[k \neq 2^n \cdot 3]$ It can be constructed from (k, k, λ) strong DF in G such that

$$\Delta C_1 \cup \cdots \cup \Delta C_s = \lambda G \text{ and } \sum_{x \in C_i} x = 0$$

□

- $v = k \cdot p^n$, is huge, p prime divisor of k

Theorem (Buratti, A.N., 2023)

Let $k \geq 3$, $k \not\equiv 2 \pmod 4$ and $k \neq 2^n \cdot 3 \geq 12$.
There are infinitely many values of v for which there exists a super-regular $(v, k, 1)$ design.

Constructing examples is computationally hard!

k	3	4	5
	$\mathrm{AG}_1(n,3)$	$\mathrm{AG}_1(n,4)$	$\mathrm{AG}_1(n,5)$

k	6	7	8	9	10
	$2^1 \cdot 3$	$\mathrm{AG}_1(n,7)$	$\mathrm{AG}_1(n,8)$	$\mathrm{AG}_1(n,9)$	$2 \pmod 4$

k	11	12	13	14	15
	$\mathrm{AG}_1(n,11)$	$2^2 \cdot 3$	$\mathrm{AG}_1(n,13)$	$2 \pmod 4$	?

- $v = 15 \cdot 5^n, n \geq 10^7$

[Caggegi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Al. str.	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$		✓		$\mathsf{AG}_1(n, p^m)$
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	$\mathsf{PG}_1(n-1, 2)$
$([2]_q, q+1, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			$\mathsf{PG}_1(2, q)$

New examples [Buratti, A.N., 2023, 202?]

Parameters	Group	Strongly	Strictly	Al. str.	Description
$(5^3, 5, 1)$	$\mathbb{F}_{5^3}$		✓		not isomorphic to $\mathsf{AG}_1(3, 5)$
$(7^3, 7, 1)$	$\mathbb{F}_{7^3}$		✓		not isomorphic to $\mathsf{AG}_1(3, 7)$
$(p^n, p, 1)$	$\mathbb{F}_{p^n}$		✓		$p \in \{5, 7\}$, $n \geq 3$, not isomorphic to $\mathsf{AG}_1(n, p)$
$([n+1]_q, [2]_q, 1)$	$\mathbb{Z}_q^{[n+1]_q}$	✓			$\mathsf{PG}_1(n, q)$
$([n+1]_q, [2]_q, 1)$	$\mathbb{F}_q^{n+1}$				$\mathsf{PG}_1(n, q)$
$(kq^n, k, 1)$	$G \times \mathbb{F}_q$		✓		$k \not\equiv 2 \pmod 4$, $k \neq 2^3 \geq 12$
$(124, 4, 1)$	$\mathbb{Z}_{124}$		✓		sporadic

New results on additive designs

[Buratti, A.N., 2023, 202?]

Parameters	Group	Strongly	Strictly	Al. str.	Description
(q^n, kq, μ)	$\mathbb{F}_{q^n}$		✓		k-parallel designs
$(p^n, p, 1)$	$\mathbb{F}_{p^n}$		✓		$p \in \{5, 7\}$, $n \geq 3$, not isomorphic to AG$_1(n, p)$
$(kq^n, k, 1)$	$G \times \mathbb{F}_q$		✓		$k \not\equiv 2 \pmod 4$, $k \neq 2^3 \geq 12$
$([n+1]_q, [d+1]_q, \lambda)$	$\mathbb{Z}_{q^d}^{[n+1]_q}$	✓			PG$_d(n, q)$
$([n+1]_q, [d+1]_q, \lambda)$	$\mathbb{F}_q^{n+1}$				PG$_d(n, q)$
(v, k, λ)	$\mathbb{Z}_{k-\lambda}^v$	✓			symmetric design
(v, k, λ)	$\mathbb{Z}_p^t$				cyclic symmetric design, p a prime dividing $k-\lambda$ but not v, $t = ord_v(p)$.
$(4\lambda+3, 2\lambda+1, \lambda)$	$\mathbb{Z}_p^t$				Paley design, $v = 4\lambda + 3$ prime, p prime divisor of $\lambda + 1$, $t = ord_v(p)$
$(4\lambda+3, 2\lambda+1, \lambda)$	$\mathbb{Z}_2^t$			✓	$v = 2^t - 1$ is a Mersenne prime
$(124, 4, 1)$	$\mathbb{Z}_{124}$		✓		sporadic

Thank you for your attention!

- A.N. The first example of a simple 2-(81, 6, 2) design. Examples and Counterexamples, 1 (2021)
- A.N., M. Buratti, Super-regular Steiner 2-designs. Finite Fields and Their Applications Volume 85, 102116 (2023)
- A.N., M. Buratti, Additivity of symmetric and subspace designs, arXiv:2307.08134