The Hadamard maximal determinant problem

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Positive Definite Matrices

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- G *is* *(Hermitian) positive definite*.

- We always use the standard inner product in this talk.
- The geometric interpretation allows us to e.g. invoke the Cauchy-Schwarz Inequality for linearly independent vectors:

$$\langle m_i, m_j \rangle^2 < \langle m_i, m_i \rangle \langle m_j, m_j \rangle \to g_{i,i} g_{j,j} - g_{i,j} g_{j,i} > 0 \,.$$

So all principal 2×2 minors of a positive definite matrix are positive.

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- Since positive definiteness is a property invariant under simultaneous permutation of rows/columns, all principal minors are positive.
- This is a higher-dimensional analogue of Cauchy-Schwarz.

Corollary

Suppose that G_k is (Hermitian) positive definite. Then

$$\det(G_k) \leq g_{k,k} \det(G_{k-1}) \,.$$

where G_{k-1} is the leading minor of size $k-1$.

$$\det\begin{pmatrix} g_{1,1} & \cdots & g_{1,k-1} & g_{1,k} \\ g_{2,1} & \cdots & g_{2,k-1} & g_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ g_{k-1,1} & \cdots & g_{k-1,k-1} & g_{k-1,k} \\ g_{k,1} & \cdots & g_{k,k-1} & g_{k,k} \end{pmatrix} = \det\begin{pmatrix} g_{1,1} & \cdots & g_{1,k-1} & g_{1,k} \\ g_{2,1} & \cdots & g_{2,k-1} & g_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ g_{k-1,1} & \cdots & g_{k-1,k-1} & g_{k-1,k} \\ 0 & \cdots & 0 & g_{k,k} \end{pmatrix} + \det\begin{pmatrix} g_{1,1} & \cdots & g_{1,k-1} & g_{1,k} \\ g_{2,1} & \cdots & g_{2,k-1} & g_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ g_{k-1,1} & \cdots & g_{k-1,k-1} & g_{k-1,k} \\ g_{k,1} & \cdots & g_{k,k-1} & 0 \end{pmatrix}.$$

The first term has determinant $g_{k,k}\det(G_{k-1})$. The second term has a 2×2 minor

$$\det\begin{pmatrix} g_{k-1,k-1} & g_{k-1,k} \\ g_{k,k-1} & 0 \end{pmatrix} = -g_{k-1,k}g^{*}_{k-1,k} < 0 \,.$$

Fischer's Inequality

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For an $n \times n$ *Hermitian positive definite matrix* $G = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$ *the inequality* $\det(G) \leq \det(A)\det(D)$ *holds.*

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- Suppose that A is $k \times k$, and define the k^{th} adjugate of M to be the $\binom{n}{k} \times \binom{n}{k}$ matrix which has as entries the minors of order k.

$$G^{(k)} = \begin{pmatrix} F & f \\ f^* & \det(A) \end{pmatrix}.$$

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- Omitting some 'well-known' facts about compound matrices, $\det(F) = \det(G)^{\binom{n-1}{k-1}-1}\det(D)$ and $\det(G^{(k)}) = \det(G)^{\binom{n-1}{k-1}}$,

$$\det(G^{(k)}) = \det(G)^{\binom{n-1}{k-1}} \leq \det(G)^{\binom{n-1}{k-1}-1}\det(D)\det(A) .$$

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- Cancel common factors.

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- This is Hadamard's inequality.

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- **Question (Hadamard bounds):** Do there exist polynomial functions $c(n), C(n)$ depending on E such that

$$\frac{1}{c(n)} n^n \leq |d_{n,E}| \leq \frac{1}{C(n)} n^n?$$

Asymptotic Existence of real matrices

Theorem (Craigen-Livinskyi, 2012)

For any odd integer n there exists $t = \lceil \alpha \log_2(n) + \beta \rceil$ *such that there exists a (real) Hadamard matrix of order* $2^t n$*. One can take* $\alpha = 1/5$ *and* $\beta = 13$*.*

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- Proof is via signed group weighing matrices and zero-correlation sequences.
- Corollary comes from estimating the difference between the orders of matrices constructed for n and $n+2$.
- Independent of existence of primes close to n.

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Suppose that $n \equiv 2 \mod 4$ *and that* $E = \{\pm 1\}$*. Then* $d_{n,E} \leq \sqrt{2n-2}(n-2)^{n-2/2}$*.*

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- Bound attained if there exist circulant A, B satisfying $AA^\top + BB^\top = (n-2)I + 2J$.

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- If $n \equiv 3 \mod 4$ the optimal Gram matrices are not known, but the determinant is bounded above by $0.6545 n^{n/2}$. The bound is not known to ever be sharp. An infinite family attaining ~ 0.48 of the bound exists when $n = 2q^2 + 2q + 3$ for odd prime power q.

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- Survey: Browne, Egan Hegarty & Ó C. *The Hadamard Maximal Determinant Problem, EJC, 2021*.

Upper bounds, real case, overview

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Upper Bound $\times n^{n/2}$	1	0.857	0.735	0.654
Best Family	1	0.857	0.735	**0.314**
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Corollary

For every congruence class $\mod 4$, *the Hadamard bound is tight (infinitely often) up to a constant factor* $C \geq 0.314$.

Lower bounds from the probabilistic method

- Let M be a matrix with entries in $\{\pm 1\}$ chosen uniformly at random. Recall the Laplace expansion of the determinant: $\sum_{\sigma \in S_n} \chi(\sigma) \prod M_{i,i^\sigma}$ where χ is the alternating character.

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- By Stirling's approximation, $\log(n!) \sim n \log n - n - \Theta(\log n)$ while $\log(n^n) = n \log n$. So a random Gram determinant is (only) a factor e^{-n} smaller than the Hadamard bound.

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- This beats Cohn: $|d_n| \geq n^{n/2} e^{-0.5n}$ vs $|d_n| \geq n^{n/2} e^{-0.62n}$.

Lower bounds from neighbouring Hadamard matrices

Proposition

Let H be a Hadamard matrix of order $n-1$. Then $\det(M) = \det(H)(1 + n^{-1}\sum_{i,j} h_{ij})$, *where*

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- Apply this result to M, observing that $\mathbf{1}^\top H \mathbf{1} = \sum_{i,j} h_{ij}$. The maximal *excess* of a Hadamard matrix of order n is $n\sqrt{n}$ with equality when all row-sums are equal.

Lower bounds from neighbouring Hadamard matrices

Proposition

Let H be a Hadamard matrix of order $n-1$. Then $\det(M) = \det(H)(1 + n^{-1}\sum_{i,j} h_{ij})$, *where*

$$M = \begin{pmatrix} H & \mathbf{1} \\ -\mathbf{1}^\top & 1 \end{pmatrix}.$$

- Schur complement: For any block matrix in which A is invertible,

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- $1/\sqrt{2}$ of the Barba bound, or 0.61 of the Hadamard bound.

Lower bounds from neighbouring Hadamard matrices

Theorem (Brent-Osborn-Smith)

Let H be a Hadamard matrix of order $n-d$. Then $M = \begin{pmatrix} H & R \\ D & S \end{pmatrix}$ *satisfies* $\det(M) \geq n^{n/2}\left(\frac{2}{\pi e}\right)^{d/2}\left(1 - d^2\sqrt{\pi(2n-2d)^{-1}}\right)$.

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- If the Hadamard conjecture holds, then the maximal determinant at order n is at least $0.11n^{n/2}$ for all n.

Lower bounds, real case, overview

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- With the strongest possible number theoretic conjectures, lower bounds are still (conjecturally) **super-polynomially** bad.

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- **Question:** Do there exist families of near-optimal matrices at odd orders?

Go raibh maith agaibh!