On r-general sets in finite projective spaces

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$\mathcal{X}$ is *complete* or *maximal* if it is maximal w.r.t. set theoretical inclusion

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parity check matrix of $\mathcal{C}$: generator matrix of $\mathcal{C}^\perp$

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r-general set $\mathcal{X}$ of $\mathrm{PG}(n, q)$ of size m	$\longleftrightarrow$	*columns of a parity check matrix H of an $[m, m-n-1]_q$ code $\mathcal{C}$*

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complete r-general set of $\mathrm{PG}(n,q)$ *of size* m	$\implies$	$[m, m-n-1]_q$ *code* $\mathcal{C}$, $\rho(\mathcal{C}) = r-1$, $d(\mathcal{C}) = r+1$.

Statistics

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"maximum number of factors that may be accommodated in constructing a design with a given constant block size and a given value of r, such that all effects involving r or less factors are confounded"

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Any point outside $\mathcal{X}$ lies on at most one $(r-2)/2$-space spanned by points of $\mathcal{X}$

$$|\mathcal{X}| + \binom{|\mathcal{X}|}{2}(q-1) + \cdots + \binom{|\mathcal{X}|}{\frac{r}{2}}(q-1)^{\frac{r-2}{2}} \leq \frac{q^{n+1}-1}{q-1}$$

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Problem: find large caps in $\mathrm{PG}(n,q)$, $n \geq 4$, $q > 2$.

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Re: Maximum size of pointsets in PG(5, 3), no four in a plane

RH Ray Hill <R.Hill@salford.ac.uk>
A: Francesco Pavese

Oggi alle 17:10

Dear Francesco

Many thanks for your very interesting email. I think this is the only communication I have received about my 1978 paper in the last 45 years!

Having now looked again at my "proof" of Theorem 4.7, I see that it relied on an extensive calculation not included in the text. I no longer have the details of this calculation, and there seems little point in trying to reproduce it, as your construction clearly shows that it must have contained an error.

Many thanks again, and good luck with all your future research.

Best wishes
Ray Hill

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I. Czerwinski, A. Pott, Sidon sets, sum-free sets and linear codes, 202?.

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Conjecture

$$M_3(4d-1,2) = 2^{2d} + 1$$

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$$M_3(4,q) \geq \begin{cases} q + \lceil 2\sqrt{q} \rceil & q = p^r, r \geq 3 \text{ odd}, \ p | \lceil 2\sqrt{q} \rceil, \\ q + \lceil 2\sqrt{q} \rceil + 1 & \text{otherwise.} \end{cases}$$

There is a transitive 4-general set in $\mathrm{PG}(5,q)$, $q \equiv 1 \pmod 3$, of size $3(q+1)$ that is the union of three twisted cubic.

$$M_3(5,5) \geq 28$$
$$\{(1, x, y, 2xy, x^2+2y^2, x^3+x^2y+xy^2-y^3) \mid x, y \in \mathbb{F}_5\}$$
$$\cup\{(0,0,0,1,0,0), (0,0,0,1,2,0), (0,0,0,0,0,1)\}.$$

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J.H. Kim & V.H. Vu 2003

$$T_2(2,q) \leq \sqrt{2q}\log^c q$$

M. Giulietti 2007

$$T_2(n,q) \leq \begin{cases} \frac{1}{2}q^{\frac{n}{2}} + 3\left(q^{\frac{n-2}{2}} + \cdots + q\right) + 2, & n, q \text{ even}, \\ \frac{5}{2}q^{\frac{n-1}{2}} + 3\left(q^{\frac{n-3}{2}} + \cdots + q\right) + 2, & n \text{ odd}, q \text{ even} \end{cases}$$

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D. Bartoli, M. Giulietti, G. Marino, O. Polverino 2018

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A. Cossidente, B. Csajbók, G. Marino, F. P. 2023

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Problem: find complete caps of $\mathrm{PG}(n,q)$ of small size

order of magnitude $cq^{\frac{n-1}{2}}$

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$$\boxed{\frac{4^d-1}{3} \leq M_3(2d-1,4) \leq \frac{\sqrt{8 \cdot 4^{2d} - 7} + 1}{6}.}$$

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$$\alpha_i \in \mathbb{F}_{q^d} \setminus \{0\},\ 1 \leq i \leq \tfrac{d-1}{2}$$

$\mathcal{V}_{\alpha_1,\ldots,\alpha_n}$ *are Veronese varieties*

A geometric description

Veronese variety of $\mathrm{PG}(W)$*: locus of the zeros of all determinants of* 2×2 *submatrices of* $M(a_0, \dots, a_{\frac{d-1}{2}})$

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$\tilde{\mathcal{V}}_\omega$ *projection of* $\mathcal{V}_{\omega,\alpha_2,\ldots,\alpha_{\frac{d-1}{2}}}$ *from* $\langle \tilde{\Pi}_2, \ldots, \tilde{\Pi}_{\frac{d-1}{2}} \rangle$ *onto* $\langle \tilde{\Pi}_0, \tilde{\Pi}_1 \rangle$

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Problem: are there complete 4-general sets in $\mathrm{PG}(n, q)$ of small size?

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order of magnitude $cq^{\frac{n-2}{3}}$

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THANK YOU