On (non)symmetric association schemes and associated family of graphs

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Rijeka, Tuesday, 4th of July, 2023.

Outline

1. Some definitions and basic results
   - Basic notation
   - Commutative association scheme
   - Our problem
2. The distance-faithful intersection diagram
   - Equitable partition with $d+1$ cells
3. Three class association schemes

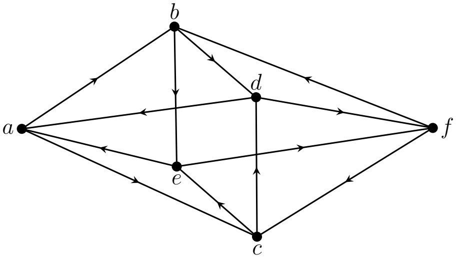
Basic notation

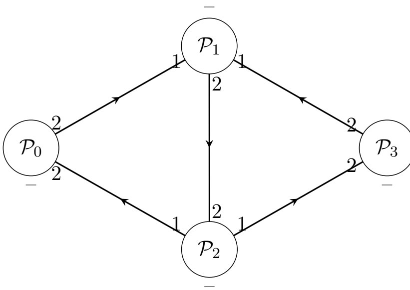
Some notation.

Γ – (strongly) connected (directed) simple graph.
X – vertex set of Γ.
$\partial(x, y)$ – distance between $x, y \in X$.
$D = \max\{\partial(x, y) \,|\, x, y \in X\}$ – diameter of Γ.
$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$.
$\Gamma_1^{\rightarrow}(x) = \{z \mid (x, z) \in \boldsymbol{E}(\Gamma)\}$.
$\Gamma_1^{\leftarrow}(x) = \{z \mid (z, x) \in \boldsymbol{E}(\Gamma)\}$.

Example of equitable distance-faithful partition

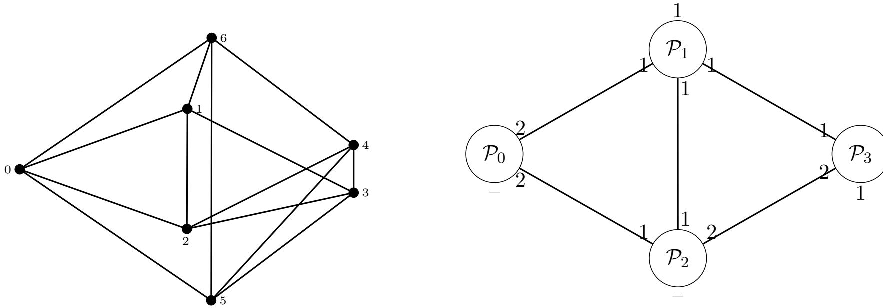
Directed graph Γ of diameter 3 and the intersection diagram of an equitable distance-faithful partition $\Pi_a = \{\mathcal{P}_0 = \{a\}, \mathcal{P}_1 = \{b, c\}, \mathcal{P}_2 = \{d, e\}, \mathcal{P}_3 = \{f\}\}$ of Γ (around vertex a).





Example of equitable distance-faithful partition (cont.)

Undirected graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_7; \{1,2\})$ of diameter 2 and the intersection diagram of an equitable distance-faithful partition of Γ (around vertex 0). The adjacency matrix of this graph generates a symmetric 3-class association scheme.



Commutative association scheme

Let X denote a finite set and $\mathrm{Mat}_X(\mathbb{C})$ the set of complex matrices with rows and columns indexed by X. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ denote a set of cardinality $d+1$ of nonempty subsets of $X \times X$. The elements of the set $\mathcal{R}$ are called *relations* (or *classes*) on X. For each integer i $(0 \le i \le d)$, let $A_i \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of the graph (X, R_i) (directed, in general). The pair $\mathfrak{X} = (X, \mathcal{R})$ is a *commutative d-class association scheme* (or a *d-class scheme* for short) if

Commutative association scheme (con.)

(AS1) $A_0 = I$, the identity matrix.

(AS2) $\sum_{i=0}^{d} A_i = J$, the all-ones matrix.

(AS3) $A_i^{\top} \in \{A_0, A_1, \ldots, A_d\}$ for $0 \leq i \leq d$.

(AS4) $A_i A_j$ is a linear combination of $A_0, A_1, \ldots, A_d$ for $0 \leq i, j \leq d$ (i.e., for every i, j $(0 \leq i, j \leq d)$ there exist *intersection numbers* p_{ij}^h, $0 \leq h \leq d$, such that $A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$).

(AS5) $A_i A_j = A_j A_i$ for every i, j $(0 \leq i, j \leq d)$ (i.e., for the intersection numbers p_{ij}^h, $0 \leq i, j, h \leq d$, from (AS4) we have that $p_{ij}^h = p_{ji}^h$).

Commutative association scheme (con.)

By $(\mathrm{AS1})$–$(\mathrm{AS5})$ the vector space $\mathcal{M} = \mathrm{span}\{A_0, A_1, \ldots, A_d\}$ is a commutative algebra; we call it the *Bose–Mesner algebra* of $\mathfrak{X}$. The set of $(0, 1)$-matrices $\{A_0, A_1, \ldots, A_d\}$ is linearly independent by $(\mathrm{AS2})$ and thus forms a basis of $\mathcal{M}$. We say that $\mathfrak{X}$ is *symmetric* if the A_i's are symmetric matrices.

Problem

In this talk we study the following problem.

Problem

Can the Bose–Mesner algebra $\mathcal{M}$ of every commutative d-class association scheme $\mathfrak{X}$ (which is not necessarily symmetric) be generated by a 01-matrix A? With other words, for a given $\mathfrak{X}$ can we find a 01-matrix A such that $\mathcal{M} = (\langle A \rangle, +, \cdot)$? Moreover, since such a matrix A is the adjacency matrix of some (directed) graph Γ, can we describe the combinatorial structure of Γ? The vice-versa question is also of importance, i.e., what combinatorial structure does a (directed) graph need to have so that its adjacency matrix will generate the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X}$?

Some of my co-authors, me and part of the team, Škocjanska jama, Slovenija, January 2023

Lemma 1

Lemma

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$ with adjacency matrices $\{A_i\}_{i=0}^d$. For a given $x \in X$ we define the partition $\Pi_x = \{\mathcal{P}_0(x), \mathcal{P}_1(x), \ldots, \mathcal{P}_d(x)\}$ of X in the following way

$$\mathcal{P}_i(x) = \{z \mid (A_i)_{xz} = 1\} \qquad (0 \le i \le d).$$

Let A denote arbitrary 01-matrix in $\mathcal{M}$, and consider (directed) graph $\Gamma = \Gamma(A)$. If Γ is (strongly) connected (directed) graph then in Γ all vertices in $\mathcal{P}_i(x)$ are at the same distance from x.

Lemma 2

Lemma

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$ with the adjacency matrices $\{A_i\}_{i=0}^d$. Pick $x, y \in X$ and define the partitions $\Pi_x = \{\mathcal{P}_0(x), \mathcal{P}_1(x), \ldots, \mathcal{P}_d(x)\}$ and $\Pi_y = \{\mathcal{P}_0(y), \mathcal{P}_1(y), \ldots, \mathcal{P}_d(y)\}$ of X on the following way

$$\mathcal{P}_i(x) = \{z \mid (A_i)_{xz} = 1\}, \quad \mathcal{P}_i(y) = \{z \mid (A_i)_{yz} = 1\} \qquad (0 \le i \le d).$$

(The lemma is continue at the next slide.)

Lemma 2 (cont.)

Lemma

Let A denote arbitrary 01-matrix in $\mathcal{M}$, and consider (directed) graph $\Gamma = \Gamma(A)$. If Γ is (strongly) connected (directed) graph then for any i,j $(0 \le i,j \le d)$ there exists scalars $D_{ij}^{\rightarrow}$ such that in Γ the following hold:

$$|\Gamma_1^{\rightarrow}(z) \cap \mathcal{P}_j(x)| = D_{ij}^{\rightarrow} \qquad \text{for every } z \in \mathcal{P}_i(x)$$

and

$$|\Gamma_1^{\rightarrow}(w) \cap \mathcal{P}_j(y)| = D_{ij}^{\rightarrow} \qquad \text{for every } w \in \mathcal{P}_i(y).$$

One of the main results

Theorem

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$, and $A \in \mathcal{M}$ denote a 01-matrix. Assume that $\Gamma = \Gamma(A)$ denotes a (strongly) connected (directed) graph. Then the following hold.

(i) *For every vertex $x \in X$, there exists an x-distance-faithful intersection diagram (of an equitable partition Π_x) with $d+1$ cells.*

(ii) *The structure of the x-distance-faithful intersection diagram (of the equitable partition Π_x) from (i) does not depend on x.*

Corollary 1

Recall that we a graph is *walk-regular* if the number of closed walks of length ℓ rooted at vertex x only depends on ℓ, for each $\ell \geq 0$ (i.e., the $(A^\ell)_{xx}$ entry for every $x \in X$ only depends on ℓ).

Corollary

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$. If a (strongly) connected (directed) graph Γ 'live' in the association scheme $\mathfrak{X}$ (i.e., if the adjacency matrix A of Γ belonts to $\mathcal{M}$) then Γ is a walk-regular graph.

Corollary 2

Corollary

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a symmetric d-class association scheme $\mathfrak{X} = (X, \mathcal{R})$, and $A \in \mathcal{M}$ denote a 01-matrix. If $\Gamma = \Gamma(A)$ generate $\mathfrak{X}$ then the following hold.

(i) *For every vertex $x \in X$, there exists an x-distance-faithful intersection diagram (of an equitable partition Π_x) with $d+1$ cells.*

(ii) *The structure of the x-distance-faithful intersection diagram (of the equitable partition Π_x) from (i) does not depend on x.*

(iii) *Graph Γ do not have x-distance-faithful intersection diagram with less than $d+1$ cells (i.e., $d+1$ is the smallest number of cells for which there exists x-distance-faithful equitable partition).*

Corollary 3

Corollary

Let $\mathcal{M}$ denote the Bose–Mesner algebra of a commutative 3-class association scheme $\mathfrak{X} = (X, \mathcal{R})$, $A \in \mathcal{M}$ denote a 01-matrix and let $\Gamma = \Gamma(A)$ denote a (directed) graph of diameter D with adjacency matrix A. If Γ generates $\mathcal{M}$ then $D \in \{2, 3\}$ and Γ has the same x-distance-faithful intersection diagram around every vertex x with 4 cells. Moreover, the following hold.

(i) *If $D = 3$, then the partition $\{\Gamma_i(x)\}_{0 \le i \le 3}$ is equitable and corresponding parameters do not depend on the choice of $x \in X$.*

(Corollary is continued at the next slide.)

Corollary 3 (cont.)

Corollary

(i) *If* $D = 2$*, then exactly one of the following* (a), (b) *holds.*

  (a) *Any two adjacent vertices have a constant number of common neighbors, and the number of common neighbors of any two nonadjacent vertices takes precisely two values.*

  (b) *Any two nonadjacent vertices have a constant number of common neighbors, and the number of common neighbors of any two adjacent vertices takes precisely two values.*

One of the main results

Recall that a 3-class association schemes is amorphic, if every graph $G_i = (X, R_i)$ $(1 \leq i \leq 3)$ is strongly-regular.

Theorem

Let $\mathfrak{X}$ denote a commutative 3-class association scheme. If $\mathfrak{X}$ is not amorphic, then there exists a (strongly) connected (directed) graph $\Gamma = \Gamma(A)$ such that the following hold.

(i) *The adjacency matrix A of Γ has exactly 4 distinct eigenvalues.*

(ii) *A generates the Bose–Mesner algebra $\mathcal{M}$ of $\mathfrak{X}$.*

Moreover, the scheme $\mathfrak{X}$ is generated by a (directed) graph if and only if it is not amorphic.

Thank you

Questions?

Thank you for your attention.

The paper will be available at ArXiV in the next few days.