Colourings of Path Systems

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Definition:

For a graph G, a G-design of order v consists of a collection $\mathcal{B}$ of subgraphs of the complete graph K_v such that

- each subgraph in $\mathcal{B}$ is isomorphic to G
- the subgraphs in $\mathcal{B}$ partition the edges of K_v

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Some families of G-designs:

- Steiner triple systems, denoted STS(v)
- balanced incomplete block designs, denoted BIBD$(v, k, 1)$
- path systems

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Some families of G-designs:

- Steiner triple systems, denoted STS(v)

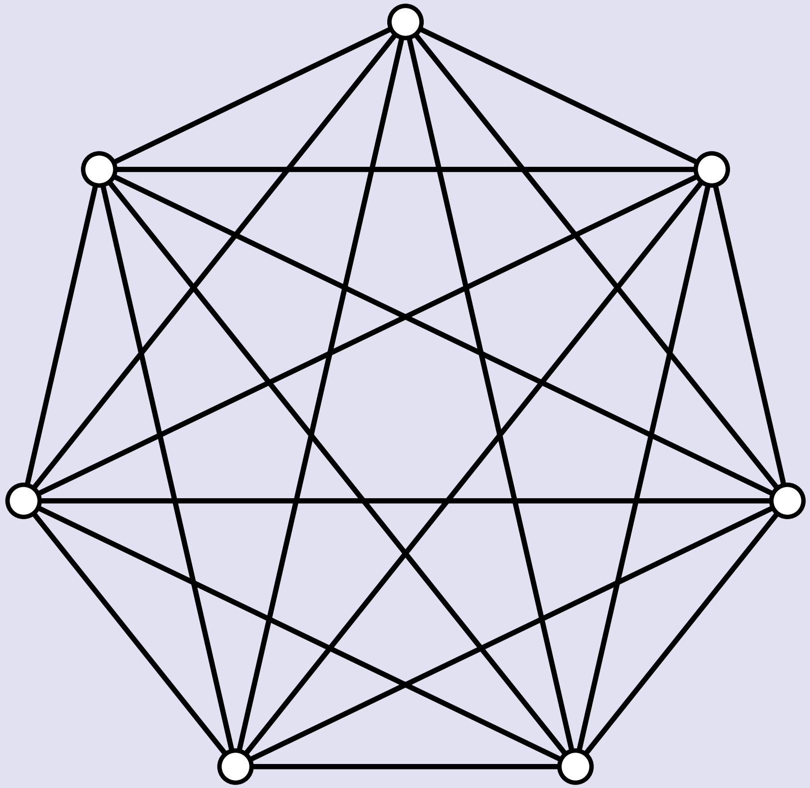
  G is a 3-cycle, namely C_3 or K_3
- balanced incomplete block designs, denoted BIBD$(v, k, 1)$

  G is a clique K_k
- path systems

  G is a path P_m

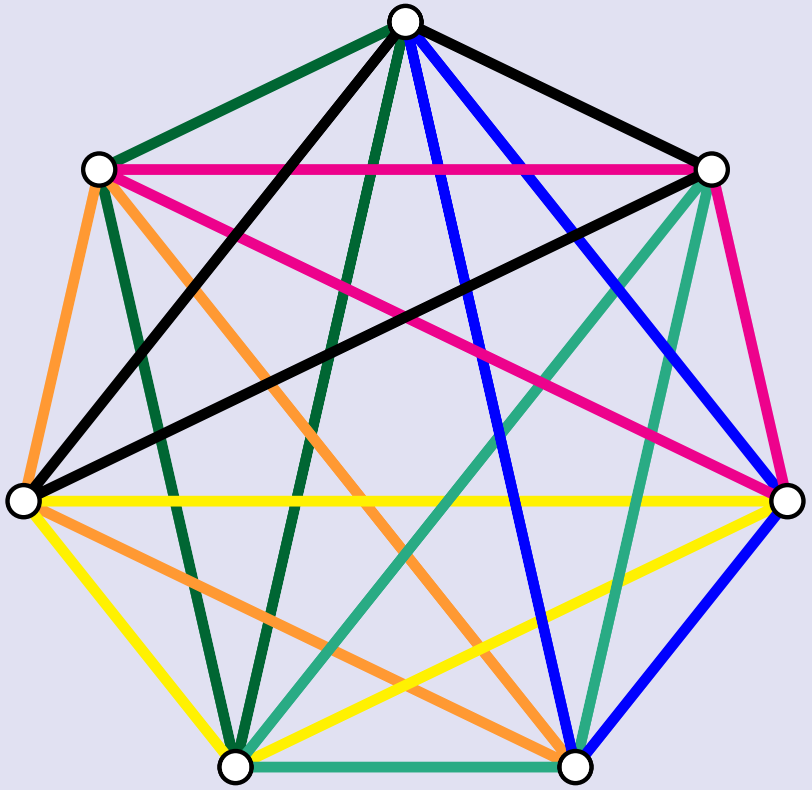
Steiner triple systems

Example: K_7



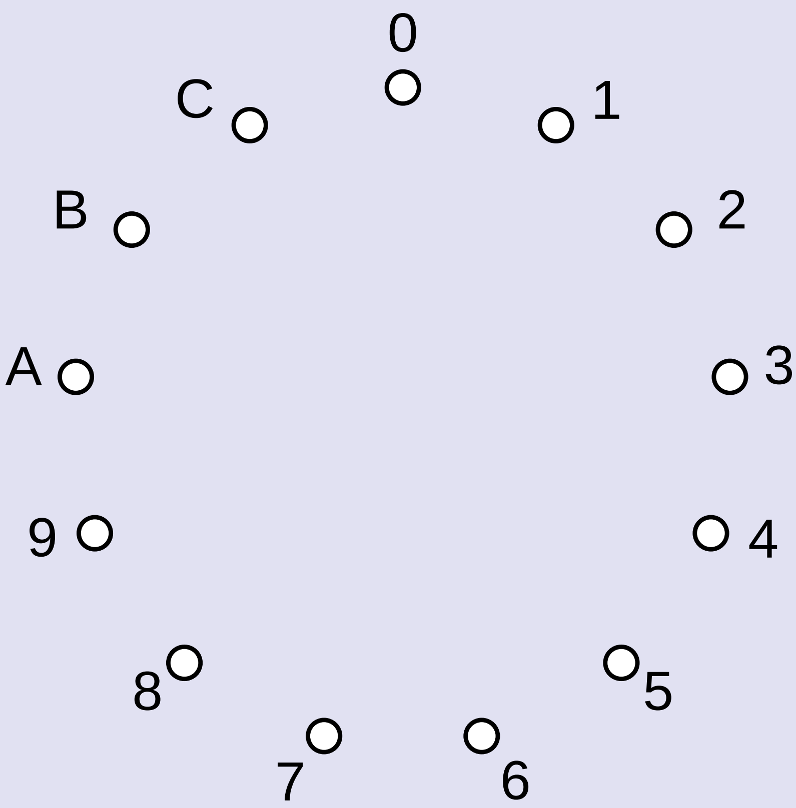
Steiner triple systems

Example: a 3-cycle decomposition of K_7



i.e., a STS(7) or a BIBD(7,3,1)

Example: A BIBD(13,4,1)



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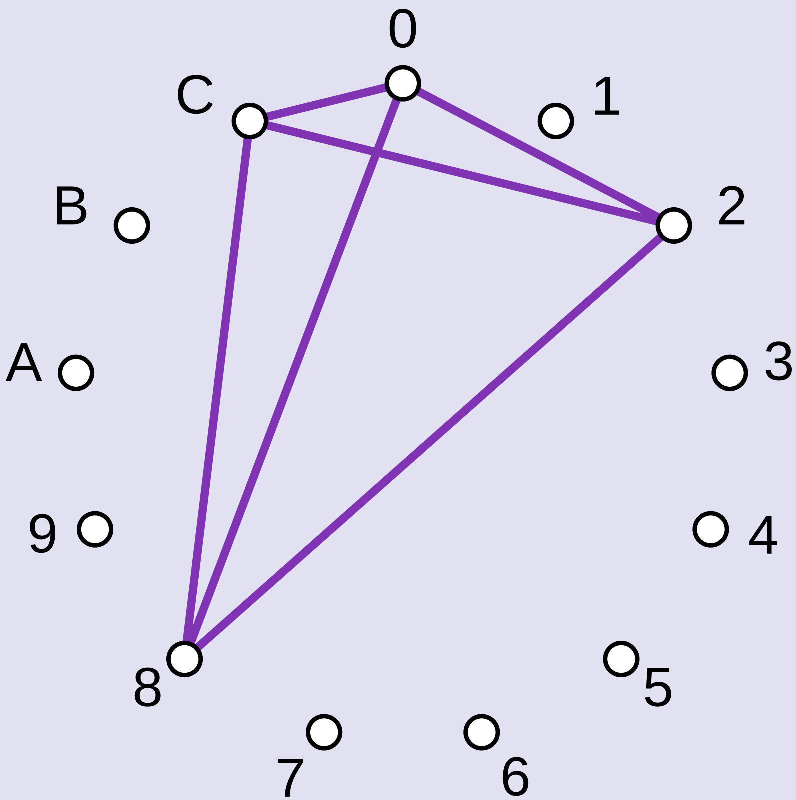

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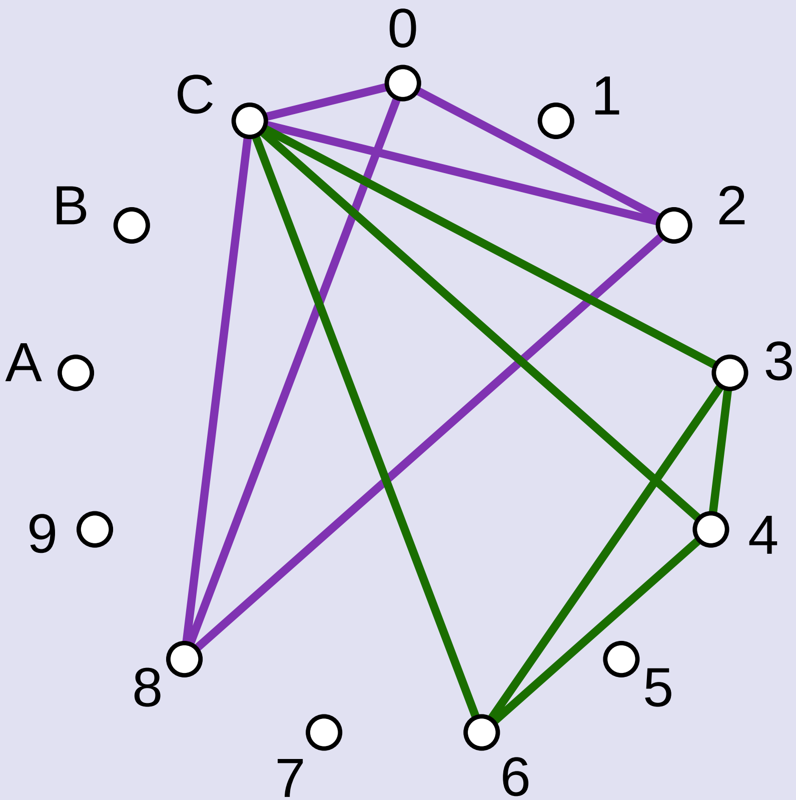

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{3,4,6,C}

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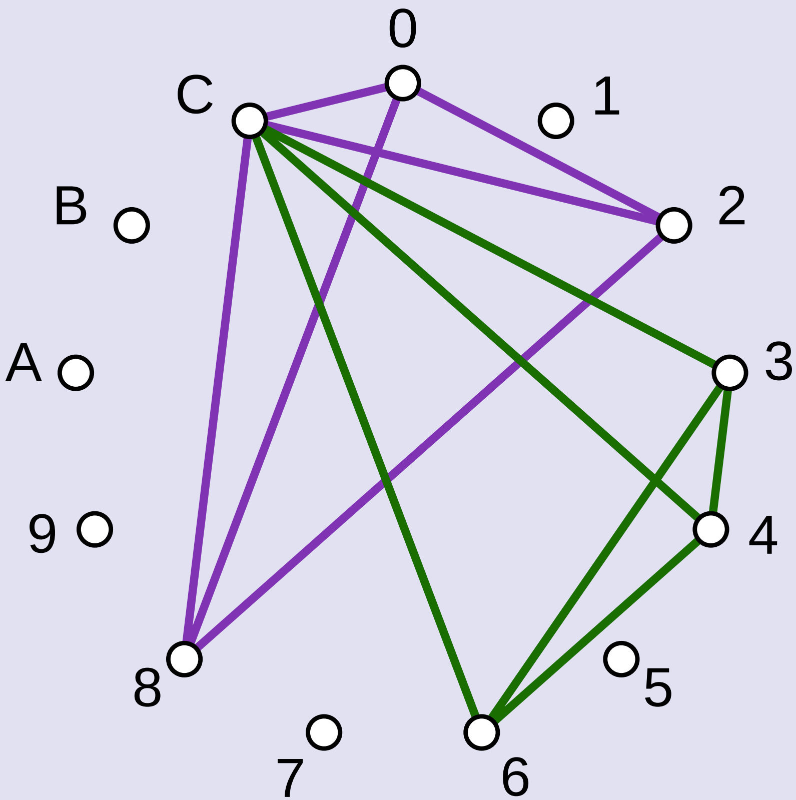

{0,1,3,9}
{0,2,8,C}
{0,4,5,7}
{0,6,A,B}
{1,2,4,A}
{1,5,6,8}
{1,7,B,C}
{2,3,5,B}
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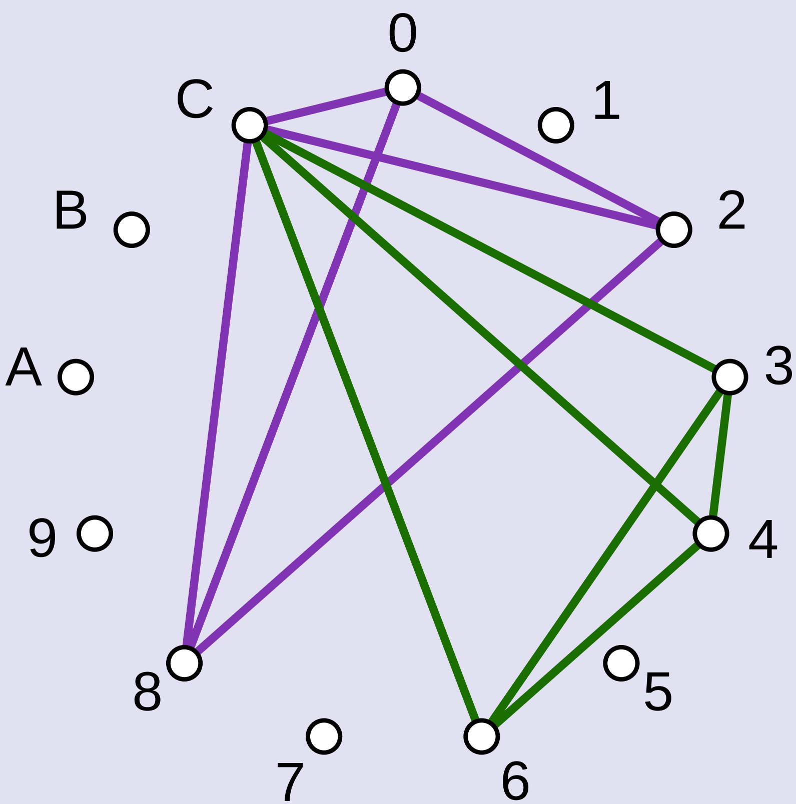

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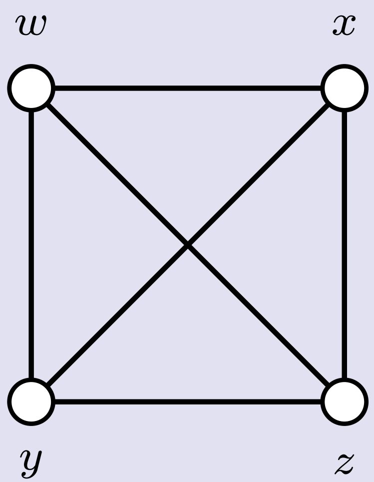
This is equivalent to a K_4-decomposition of K_{13}

Example: A P_3-decomposition of K_4



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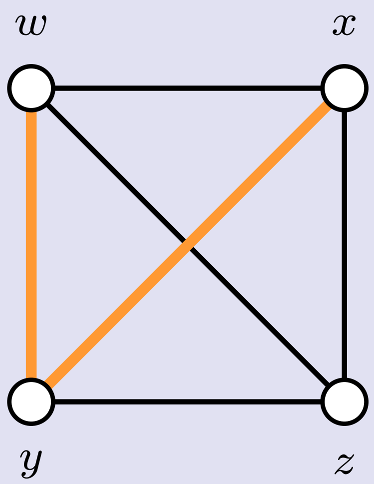

$$\mathcal{B} = \{(w, y, x)$$

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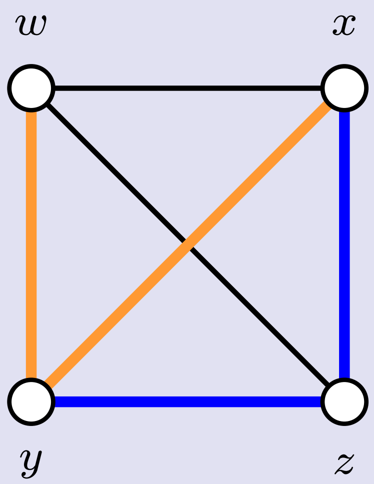

$$\mathcal{B} = \{(w, y, x), (x, z, y)$$

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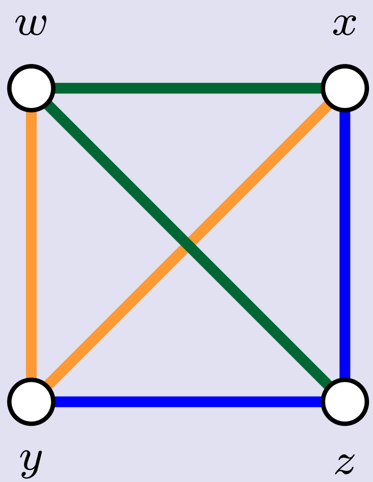

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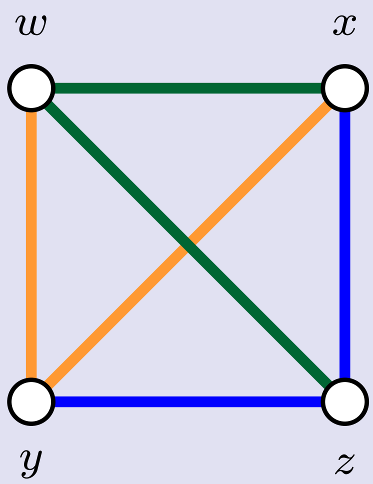

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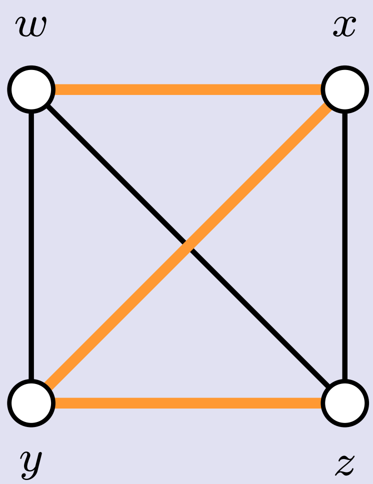

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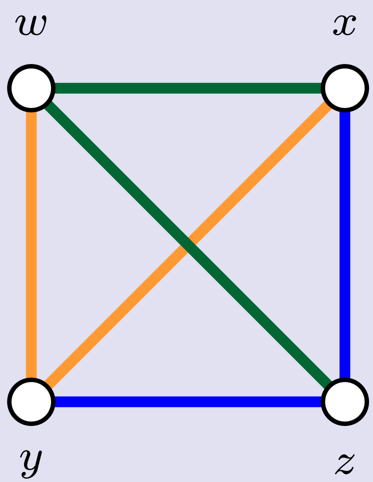

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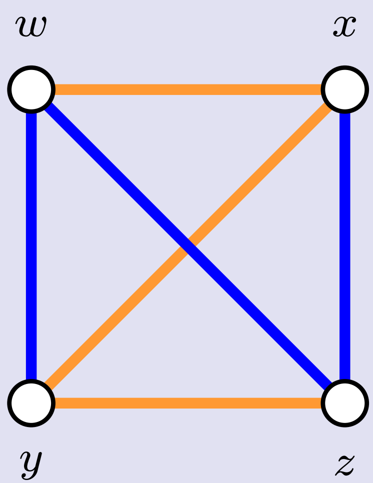

$$\mathcal{B} = \{(w, x, y, z), (x, z, w, y)\}$$

Conditions for existence:

Theorem (Tarsi, 1983)

A P_m-decomposition of K_v exists if and only if

- $m \leqslant v$
- $(m-1)$ divides $\binom{v}{2}$

Definition

Any integer v satisfying the above criteria will be called P_m-admissible or just admissible.

Resolvable Path Decompositions:

Theorem (Horton, 1985)

A resolvable P_3-decomposition of K_v exists if and only if $v \equiv 9 \pmod{12}$

Theorem (Bermond, Heinrich and Yu, 1990)

A resolvable P_m-decomposition of K_v exists if and only if

- $v \equiv 0 \pmod{m}$
- $m(v-1) \equiv 0 \pmod{2(m-1)}$

Definition:

A weak c-colouring of a design $\mathcal{D}$ consists of a partition of the points of $\mathcal{D}$ into c colour classes such that no block of $\mathcal{D}$ is monochromatic.

A design $\mathcal{D}$ is said to be c-chromatic if c is the smallest integer for which $\mathcal{D}$ admits a weak c-colouring.
Notation: we write $\chi(\mathcal{D}) = c$ if $\mathcal{D}$ is c-chromatic.

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A BIBD(13,4,1)

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This design is 2-chromatic.

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{0,1,3}	{1,2,4}	{2,3,5}
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A STS(7)

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This design is 3-colourable.
But is it 3-chromatic?

Some History – Steiner Triple Systems

- Every STS(v) with $v \geqslant 7$ requires at least 3 colours.
  (Rosa and Pelikán, 1970)
- Every STS(v) with $7 \leqslant v \leqslant 15$ is 3-chromatic.
  (Mathon, Phelps and Rosa, 1983)
- Every STS(19) is 3-chromatic. (Colbourn et al., 2010)

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- For each c there is a STS with chromatic number at least c.
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- For each $c \geqslant 3$ there is a c-chromatic STS.
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- For each $c \geqslant 3$ there is a c-chromatic STS.
  (de Brandes, Phelps and Rödl, 1982)
- There is a 4-chromatic STS(21). (Haddad, 1999)
- There is a 5-chromatic STS(63). (Fugère, Haddad and Wehlau, 1994)
- There is a 6-chromatic STS(243). (Bruen, Haddad and Wehlau, 1998)

Some History – BIBDs

- For each admissible v, i.e. $v \equiv$ 1 or 4 (mod 12), there is a 2-chromatic BIBD$(v, 4, 1)$. (Hoffman, Lindner and Phelps, 1990) (Franek, Griggs, Lindner and Rosa, 2002)
- A 3-chromatic BIBD$(v, 4, 1)$ exists if and only if $v \equiv$ 1 or 4 (mod 12) and $v \geqslant 25$. (Rodger, Wantland, Chen, Zhu, 1994)
- The obvious necessary conditions for the existence of a BIBD$(v, 4, 1)$ are asymptotically sufficient for the existence of a c-chromatic BIBD$(v, 4, 1)$. (Linek and Wantland, 1998)
- For each admissible v, i.e. $v \equiv$ 1 or 5 (mod 20), there is a 2-chromatic BIBD$(v, 5, 1)$. (Ling, 1999)

Some History – BIBDs

- For $\lambda \geqslant 2$, for each admissible v, there is a 2-chromatic BIBD$(v, 4, \lambda)$. (Hoffman, Lindner and Phelps, 1990) (Hoffman, Lindner and Phelps, 1991) (Rosa and Colbourn, 1992)

- For all integers $\lambda \geqslant 1$, $c \geqslant 2$ and $k \geqslant 3$ with $(c, k) \neq (2, 3)$, the obvious necessary conditions for the existence of a BIBD(v, k, λ) are asymptotically sufficient for the existence of a c-chromatic BIBD(v, k, λ). (Horsley and Pike, 2014)

Path Systems

Theorem (Darijani and Pike, 2023+)

For each $c \geqslant 2$ and $m \geqslant 3$ there exists a c-chromatic P_m system.

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For each $c \geqslant 2$ and $m \geqslant 3$ there exists a c-chromatic P_m system.

Proof Sketch for when m is even:

For sufficiently large v, there exists a c-chromatic BIBD$(v, m, 1)$ (by Horsley and Pike, 2014).

The graph K_{m+1} can be decomposed into Hamilton cycles (by Walecki, 1890s). Deleting one vertex from K_{m+1} yields a decomposition of K_m into P_m paths.

By decomposing each block of the BIBD$(v, m, 1)$ into P_m paths, we obtain a c-chromatic P_m system of order v.

Path Systems

Theorem (Darijani and Pike, 2023+)

For each $c \geqslant 2$ and $m \geqslant 3$ there exists a c-chromatic P_m system.

Proof Sketch for when m is odd and $m \geqslant 5$:

For sufficiently large v, there exists a c-chromatic BIBD$(v, m, 1)$ such that $v \equiv 0 \pmod{m}$ and $v - 1 \equiv 0 \pmod{2m-2}$.

The block set of this BIBD can be partitioned into pairs of blocks B and B' that share one point.

The graph K_m can be decomposed into Hamilton cycles.
By removing a particular edge from each Hamilton cycle, a set of P_m paths is obtained from B and B'.

We obtain a c-chromatic P_m system of order v.

P_4 Path Systems

Theorem (Darijani and Pike, 2023+)

For each $c \geqslant 2$, there exists a c-chromatic P_4 system of order v for all sufficiently large admissible v.

Recall: The admissible orders for a P_4 system are $v \equiv 0, 1 \pmod 3$)

Proof Sketch:

For $c = 2$, we show that there exist 2-chromatic systems for all admissible orders (by direct construction).

For $c \geqslant 3$, we show how to use a c-chromatic system of order v to construct c-chromatic systems for some of the next admissible orders. We then iterate this process, beginning with initial instances (such as those from the previous slides).

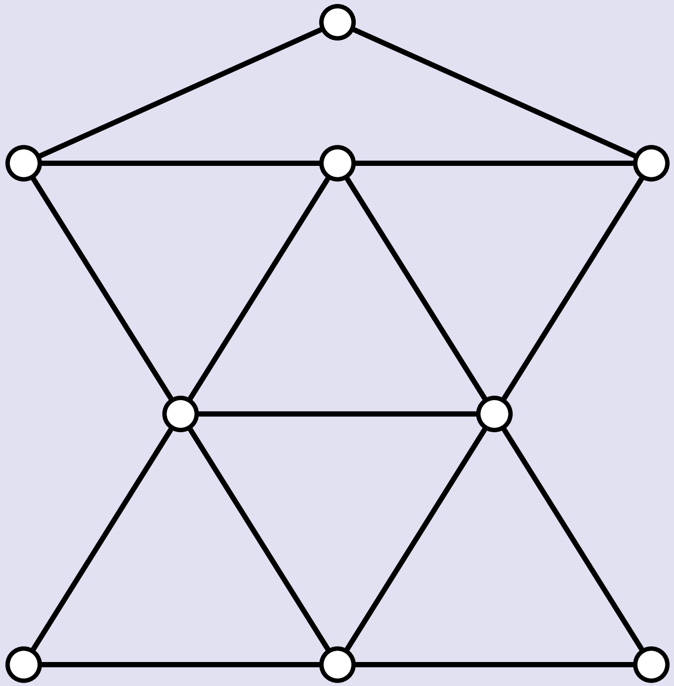
Unique Colourings

Definition:

A c-colouring of a design is unique if every c-colouring of the design has the same partition into colour classes.

Note that whenever a design is uniquely c-colourable, it must also be c-chromatic.

An easy example of a uniquely 3-colourable graph



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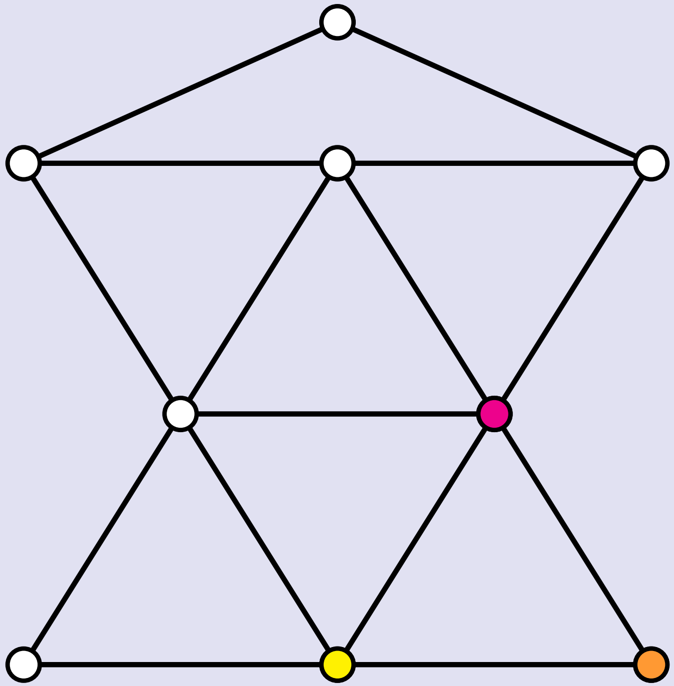
At least 3 colours are necessary for a proper colouring.



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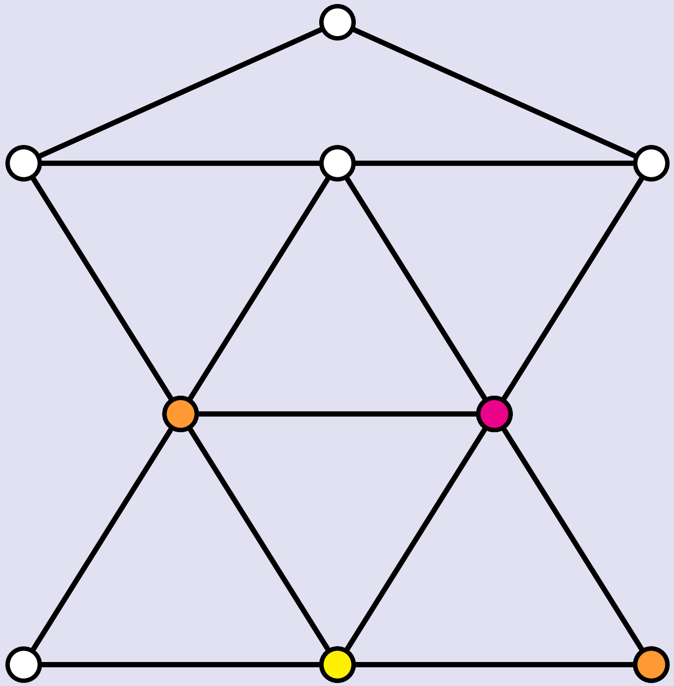

We now have no choice.

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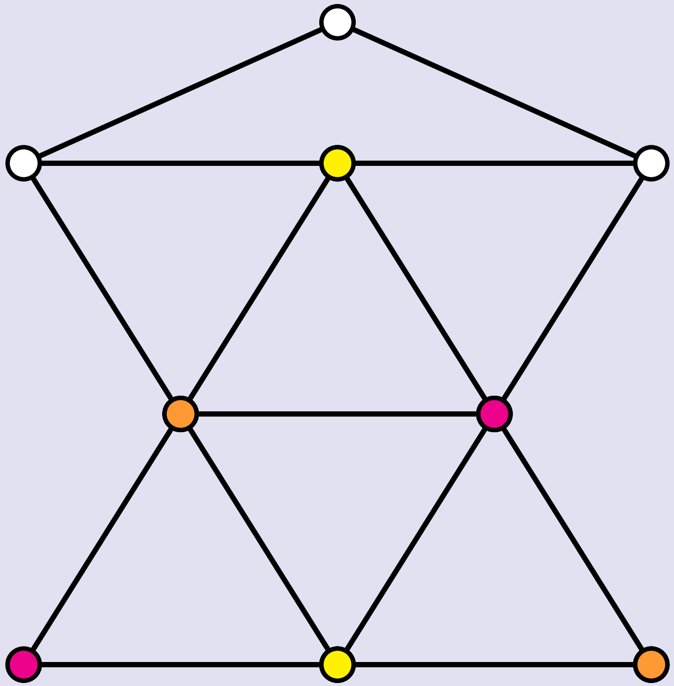

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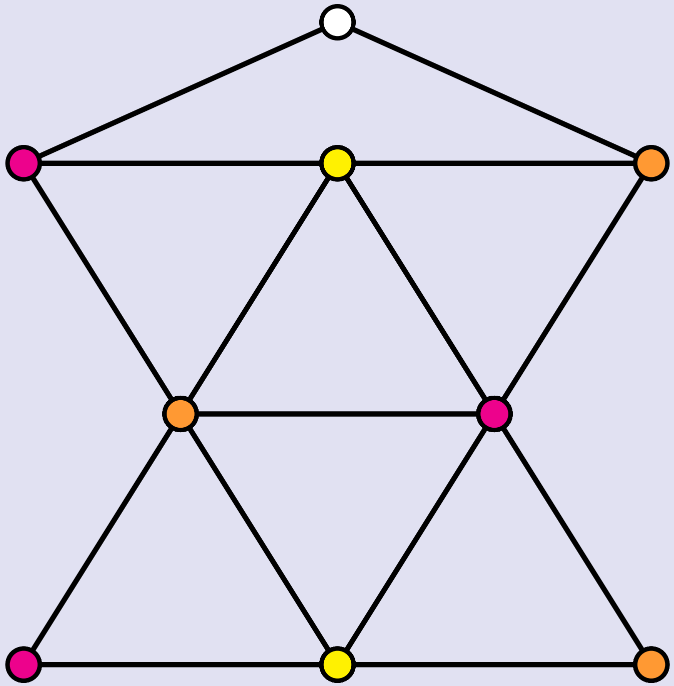

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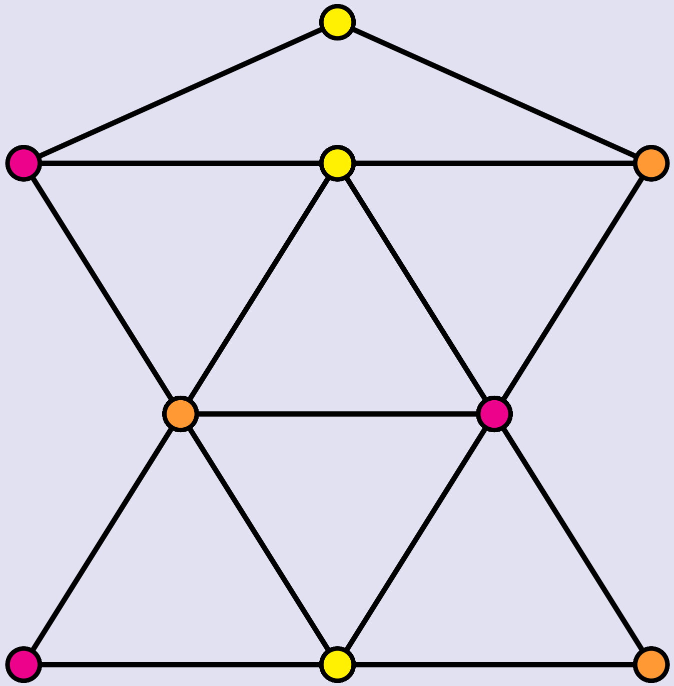

This is a unique partition into 3 colour classes.

Some History – Unique Colourings of Designs

- A uniquely 3-colourable STS(33) is observed to exist.

  (de Brandes, Phelps and Rödl, 1982)

  (Colbourn, Haddad and Linek, 1997)

- For each $v \geqslant 25$ there exists a uniquely 3-colourable STS(v)

  (Forbes, 2003)

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As far as I am aware, when $c \geqslant 4$, no examples of uniquely c-colourable STS are currently known to exist.

Uniquely c-colourable BIBD$(v, k, 1)$ with $k \geqslant 4$ are also unknown.

Uniquely c-colourable e-star systems were studied.

(Darijani and Pike, 2020)

P_4 Path Systems

Theorem (Darijani and Pike, 2023+)

There exists a uniquely 2-colourable P_4 system of order v for each admissible $v \geqslant 109$.

Uniquely 2-Colourable P_4 Systems

- We built a system $\mathcal{S}_{28}$ of order 28 in which two specific points cannot have the same colour in any 2-colouring.
- We then used $\mathcal{S}_{28}$ to build a partial system on 109 points that has a unique 2-colouring.
- We successfully completed this partial system to a full system of order 109, by adding more blocks, none of which are monochromatic.
- Finally we showed how to iteratively take a uniquely colourable system and embed it in slightly larger systems that are also uniquely colourable.

Building $\mathcal{S}_{28}$

- Suppose points 27 and 28 have the same colour, say white.

- For $i = 1, 2, \ldots, 12$, add the block $(27, 2i, 2i-1, 28)$.
  This forces one of points 1 & 2 to be black,
  one of points 3 & 4 to be black, etc.

- For distinct $i, j, \ell \in \{1, 2, \ldots, 12\}$ consider these blocks:

$$(2i-1, 2j, 2\ell, 2i) \qquad (2i-1, 2j-1, 2\ell-1, 2i)$$

$$(2i-1, 2\ell-1, 2j, 2i) \qquad (2i-1, 2\ell, 2j-1, 2i)$$

  Adding these blocks for $(i, j, \ell) = (1, 2, 3)$ forces
  points 1 & 2 to have different colours.

  Do similar for $(i, j, \ell) = (4, 5, 6), (7, 8, 9), (10, 11, 12)$
  so that points 7 & 8, 13 & 14, 19 & 20 are different too.

Building $\mathcal{S}_{28}$ (continued)

- Specifying some more blocks forces points 1,7,13,19 to be one colour and 2,8,14,20 to be the other.
- Similar steps force points 3,9,15,21 to be one colour and 4,10,16,22 to be the other.
- A monochromatic block is eventually forced.
  Hence the supposition that 27 & 28 have the same colour is contradicted.
- Demonstrating a valid 2-colouring where 27 and 28 are different is all that remains, to confirm that the system's chromatic number is not larger than 2. In fact, we exhibit a 2-colouring for which the colour classes have equal size.

Building $\mathcal{S}_{109}$

- Take four copies of $\mathcal{S}_{28}$ on point sets

$$\{1, 2, \ldots, 27, 28\} \qquad \{1', 2', \ldots, 27', 28\}$$
$$\{1'', 2'', \ldots, 27'', 28\} \qquad \{1''', 2''', \ldots, 27''', 28\}$$

- If point 28 is white, then 27, $27'$, $27''$ and $27'''$ are each black.

- Add blocks between the four copies of $\mathcal{S}_{28}$ such as:

$$(27, 1', 27'', 27') \qquad (27, 3', 27'', 27''')$$
$$(27, 1'', 27''', 27') \qquad (27', 27, 1''', 27'')$$

  Points $1'$, $3'$, $1''$ and $1'''$ are therefore white.

- Add more blocks, eventually forcing each point's colour.

- Then complete the design by adding additional blocks (none of which are monochromatic).

Future Work

- Prove that uniquely c-colourable P_4 systems exist for $c \geqslant 3$.
- Also consider P_m systems with $m \neq 4$

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- Prove that uniquely c-colourable P_4 systems exist for $c \geqslant 3$.
- Also consider P_m systems with $m \neq 4$

- Prove that uniquely c-colourable Steiner triple systems exist for $c \geqslant 4$.
- Also consider (v, k, λ)-BIBDs with $k \geqslant 4$ and m-cycle systems with $m \geqslant 4$.

Thank you.

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