A Strategy for Generating Polycyclic Configurations

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A breakthrough in the modern study of geometric configurations of points and lines came with the seminal 1990 paper [7] of Grünbaum and Rigby in which the first geometric point-line representation of a 4-configuration was constructed.

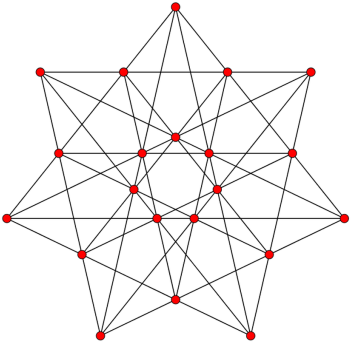
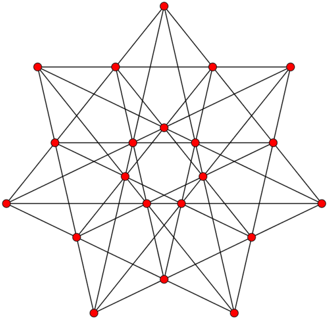


Figure: The Grünbaum–Rigby (21_4) geometric configuration, denoted by $\mathrm{GR}(21_4)$.

This (21_4) configuration was based on the work of Felix Klein [8] on his famous quartic curve, and is nowadays known as the Grünbaum–Rigby configuration.

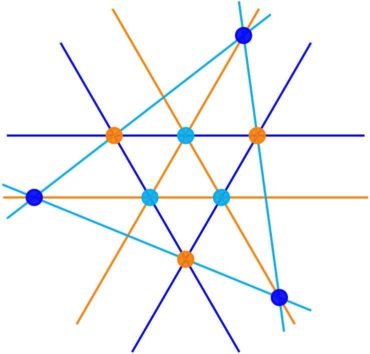


What is a geometric 4-configuration?
It is a structure composed of points and lines such that
- Each line contains 4 points.
- Each point belongs to 4 lines.

The Grünbaum–Rigby (21_4) geometric configuration, denoted by $\mathrm{GR}(21_4)$.

- Rotational symmetry.
- 3 point orbits, 3 line orbirts
- All orbits of the same size (=7)!

Rotational symmetry of a (9_3) configuration.



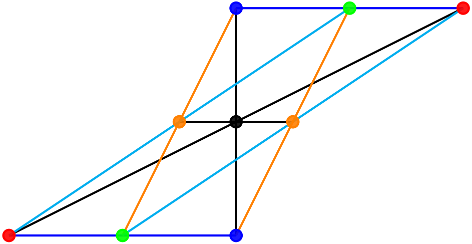
Color-coded orbits under cyclic (rotational) symmetry of this geometric 3-configuration.

- 3 lines in blue orbit,
- 3 lines in orange orbit,
- 3 lines in cyan orbit,
- 3 points in blue orbit,
- 3 points in orange orbit,
- 3 points in cyan orbit,

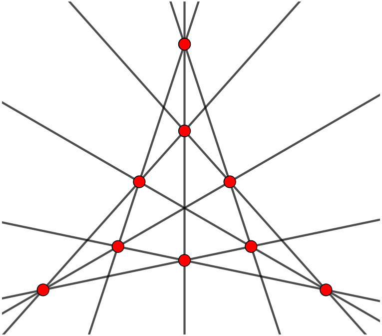
All orbits of the same size (= 3)!

Two versions of Pappus configuration (9_3).

Geometric configuration with rotational symmetry where all orbits have the same size is polycyclic configuration.



Rotational symmetry but not polycyclic!



Polycyclic version of Pappus.

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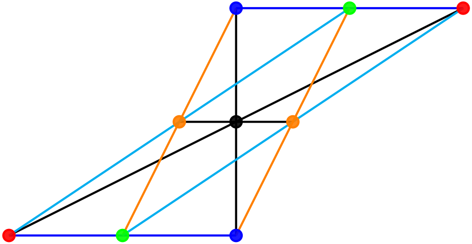

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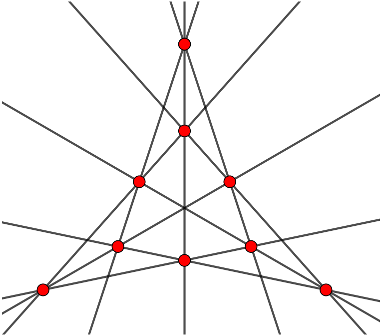
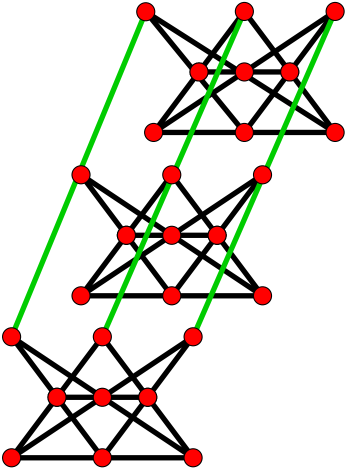


Polycyclic version of Pappus.

Later Branko Grünbaum discovered a large number of (n_4) configurations. Some of them were constructed in the spirit of $\mathrm{GR}(21_4)$, while others were constructed by various techniques from smaller ones. Nowadays only the existence of geometric (23_4) configuration is undetermined.

Parallel Switch; A Method for constructing large configurations from smaller building blocks.



• From any (m_k) configuration $\mathcal{C}$ we obtain a (km_k) configuration $\mathcal{D}$. If $\mathcal{C}$ is connected, then $\mathcal{D}$ is connected, too.

• In this example, from three copies of Pappus (9_3) a (27_3) configuration is obtained.

In 2003 Boben and Pisanski [1] initiated the theory of polycyclic configurations, having $\mathrm{GR}(21_4)$ and some other configurations from another paper (1992) of Grünbaum (co-authored by Harold Dorwart) [4] as the prime models of such configurations. This strategy was independently pursued and further developed by Grünbaum and Berman. It is closely intertwined with graph theory as well; for some details on this connection, see [9].

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• The key tool for studying configurations was the underlying combinatorial incidence structure which has an equivalent representation as a bipartite incidence graph or what Coxeter called in 1950, Levi graph (with a given vertex 2-coloring).

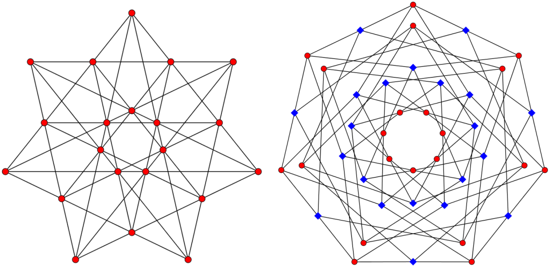


Figure: The Grünbaum–Rigby (21_4) geometric configuration and its Levi graph.

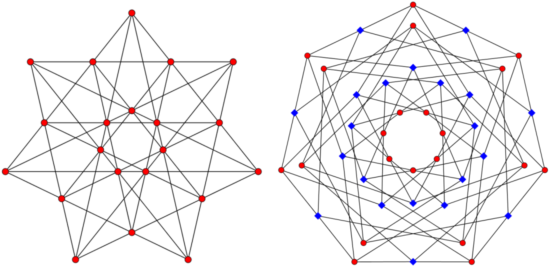


Figure: The Grünbaum–Rigby (21_4) geometric configuration and its Levi graph.

- There is an obvious geometric rotational symmetry of order 7. Hence $\mathbb{Z}_7$ acts on points and lines. The action is semiregular.
- If, in addition, we consider reflections, there is an action of dihedral group $\mathbb{D}_7$ or order 14. However, the action is not semiregular since the orbits have only size 7.
- The automorphism group of the corresponding Levi graph has order 672. The graph is vertex-transitive.
- Half (336) of the graph automorphisms respect vertex colors. They correspond to the automorphisms of the combinatorial configuration.
- Half (336) of the graph automorphisms interchange vertex colors. They correspond to self-dualities of the corresponding combinatorial configuration.

In their paper Grünbaum and Rigby conjectured that no other (n_4) configuration exists for $n \leq 21$. It was a big surprise when Grünbaum himself in 2006 disproved this conjecture [5] by constructing a (20_4) configuration, that we denote by $\mathrm{G}(20_4)$.

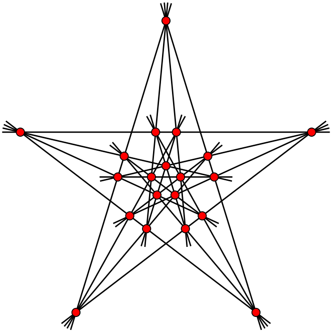


Figure: The Grünbaum (20_4) geometric configuration, denoted by $\mathrm{G}(20_4)$.

Later, Bokowski and his co-workers showed that there are exactly two distinct (18_4) configurations and proved that no geometric (19_4) configuration exists [3, 2].

Although their conjecture has been disproven, it was widely believed that the $\mathrm{GR}(21_4)$ configuration is the only geometric 4-configuration for $n = 21$ and that $\mathrm{G}(20_4)$ is the smallest geometric 4-configuration.

Several months ago, surprisingly Leah Berman constructed a new (21_4) geometric configuration, depicted in Figure 4. We denote it by $\mathrm{B}(21_4)$.

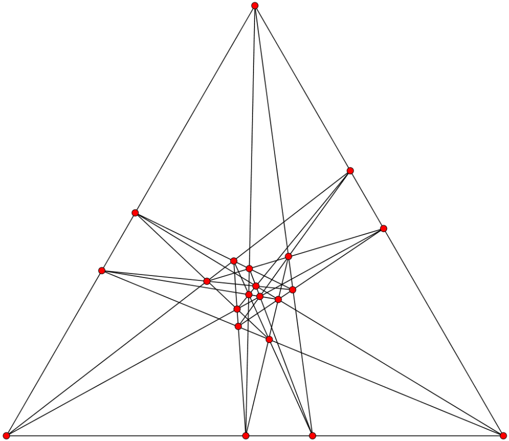
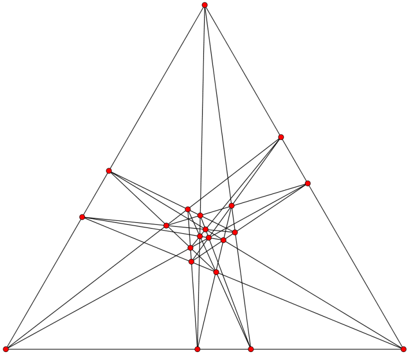


Figure: A new (21_4) geometric configuration, denoted by $\mathrm{B}(21_4)$ It has a 3-fold rotational symmetry, making the configuration polycyclic.



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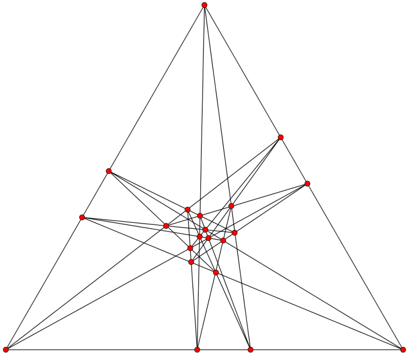
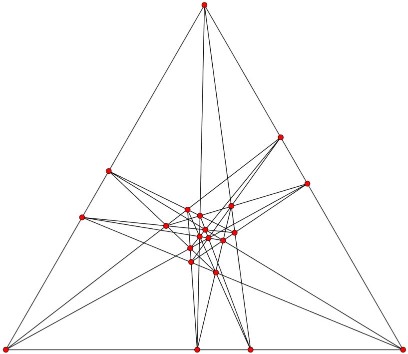


- How did she do it?
- How did she discover this fascinating 4-configuration, the second (21_4) configuration, more than 30 years after the first (21_4) configuration has been discovered?



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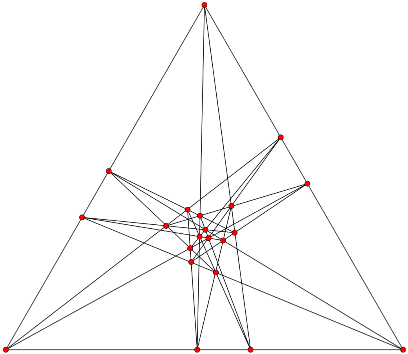

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• Let us change the angle and start with graphs.

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Note that both m and k are divisors of graph order n. While $k = 1$ is excluded, the other extreme, $k = n, m = 1$ is legitimate. The $(1, n)$-semiregular automorphism is regular. In this case the corresponding graph is a circulant graph, a Cayley graph for cyclic group $\mathbb{Z}_n$.

This justifies the following definition:

A graph G admitting a semi-regular automorphism is polycirculant. In particular, if α is an (m, k)-semiregular automorphism, the graph is (m, k)-multicirculant. Moreover, if $k = 1$, then G is a circulant graph, for $k = 2$ it is a bicirculant graph, for $k = 3$ it is a tricirculant graph, for $k = 4$ it is tetracirculant graph, etc.

Note that the Levi graph of $GR(21_4)$ is a $(6, 7)$-multicirculant while the Levi graph of $B(21_4)$ is a $(14, 3)$-multicirculant.

Let G be a graph and let α be a (m, k)-semiregular automorphism. Then the quotient G/α is well-defined and the projection $\pi : G \to G/\alpha$ is a regular covering projection. G is a $\mathbb{Z}_k$ covering graph over G/α. Note that G/α is a pregraph (may have multiple edges, loops and semi-edges.) Theory of covering graphs and voltage graphs applies.

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What about the converse? Which polycirculant graphs are Levi graphs of polycyclic configurations?

We give an answer to the above question in the combinatorial – group theoretical settings.

Every (m, k)-circulant graph G together with a $(m.k)$-semiregular automorphism α may be recovered from its quotient graph $B = G/\alpha$ by assigning appropriate voltages, i.e. elements from $\mathbb{Z}_k$ to arcs of B such that reverse arcs are assigned complementary voltages.

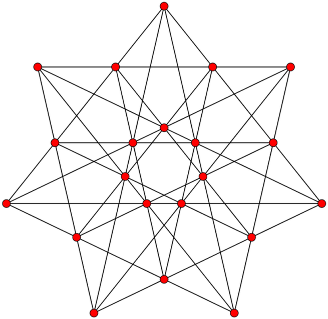
Starting with voltage graph on B the following is needed to get a Levi graph of a connected (m, k)-cyclic d-configuration.

• B is connected, regular of order k and valence d. [Otherwise the result is not a d-configuration.]

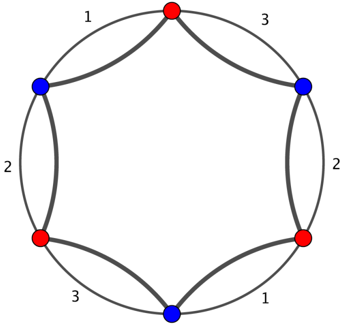
• gcd of all voltages and k is equal to 1. [Ensures connectivity]

• For any closed walk in B of length r with accumulated voltage $a \in \mathbb{Z}_m$ such that $b = \gcd(a, k)$ and $c = k/b$ we need: $rc \geq 6$. [no cycle of length < 6 in G]

Under these conditions B is a reduced Levi graph.



$GR(21_4)$



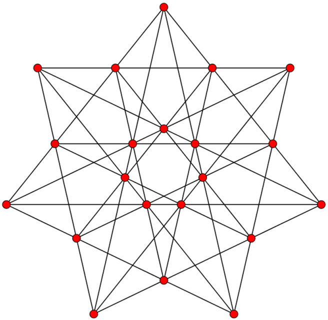
RLG of $GR(21_4)$, voltage group $\mathbb{Z}_7$

Since $21 = 7 \times 3$ we have $3 + 3$ orbits of size 7.

Gábor Gévay asked whether there exists a realization of GR configuration with $7 + 7$ orbits of size 3.

How many reduced Levi graphs does (combinatorial) GR configuration have?

- GR has 672 automorphisms.
- GR has 314 semiregular automorphisms.
- GR has 8 non-isomorphic quotient graphs.
- GR has 2 reduced Levi graphs.



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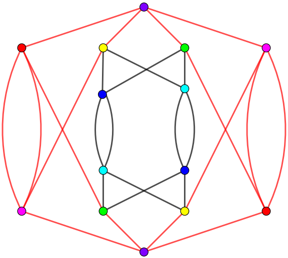
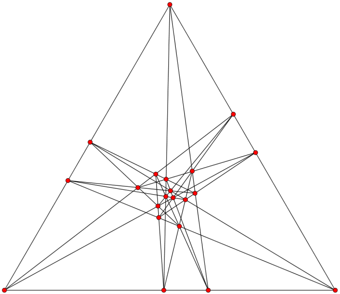


RLG of $GR(21_4)$, voltage group $\mathbb{Z}_3$

Unfortunately, this RLG has no geometric realization. (Shown by Leah Berman).



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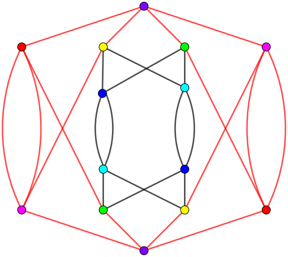


The RLG of $B(21_4)$, voltage group $\mathbb{Z}_3$

However, by changing voltages in this RLG exactly one new (21_4) configuration, namely $B(21_4)$ was constructed.

How many reduced Levi graphs does (combinatorial) B configuration have?

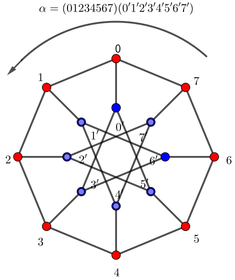
- B has 12 automorphisms.
- B has 314 semiregular automorphisms.
- B has 8 non-isomorphic quotient graphs.
- B has 1 reduced Levi graphs.

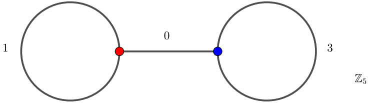
I used the same program with nauty to experiment with 3-configurations.

n	(a)	(b)	(c)	(d)	(e)
7	1	1	1	1	1
8	1	1	1	1	1
9	3	3	3	3	3
10	10	10	10	2	2
11	31	25	28	1	1
12	229	95	162	14	14
13	2036	366	1201	2	2
14	21398	1432	11415	51	45

- (a) Number of configurations
- (b) Number of self-dual configurations
- (c) Number of Levi graphs
- (d) Number of polycyclic configurations
- (e) Number of polycyclic Levi graphs (i.e. graphs admitting at least one reduced Levi graph)

The usual quotient graph of the Möbius-Kantor graph is the handcuff graph. However,





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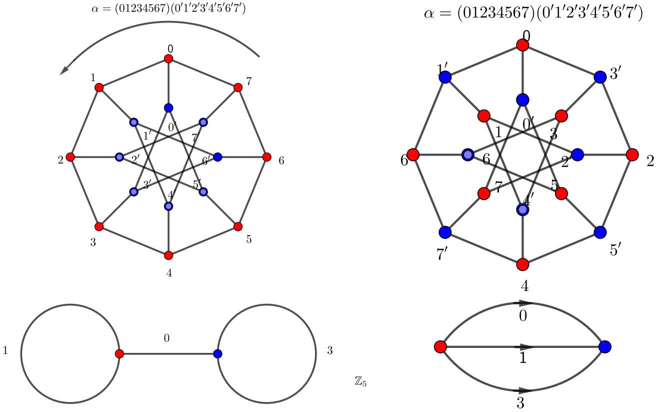

there is another $(2, 8)$ semi-regular automorphism. The corresponding quotient is the dipole alias theta graph θ_3. It is a reduced Levi graph. Hence Möbius-Kantor is (poly)cyclic!

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> Does every vertex transitive graph admit a hamilton path?

With Simona Bonvicini we transferred this problem to bicirculants. In particular, we completed classification of cubic bicirculants that are hamiltonian. Currently we are working on hamiltonicity of bicirculants of higher degrees.

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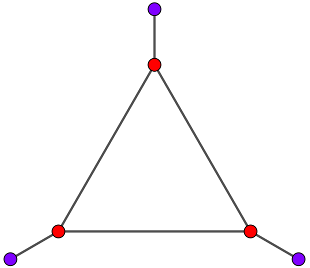

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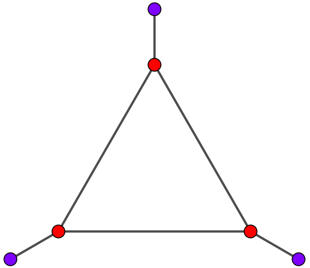
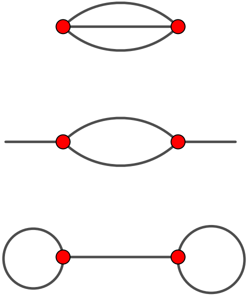
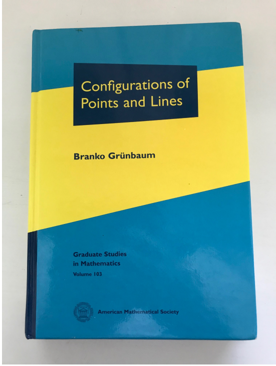


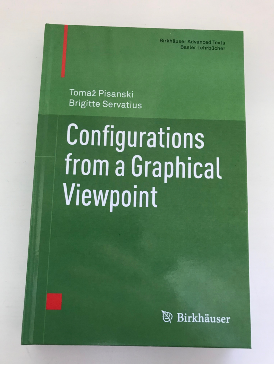
Usually we consider only regular polycirculant graphs.

Bicirculants and other polycirculants have only finitely many different quotients. For instance, there are only three types of cubic bicirculants.



- Top - Cyclic Haar graphs, Cayley graphs of dihedral group.
- Middle - Prisms and Möbius ladders.
- Bottom - I-graphs that include generalized Petersen graphs.





Two books on configurations with different emphases. One (2009) is focused on geometry the other one (2013) on algebraic graph theory.

Thank you!

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