Beyond Fibonacci cubes and Pell graphs

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(joint work with Tomislav Došlić)

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Hypercubes

Alphabet $\mathcal{T} = \{0,1\}$. Adjacency $0 \leftrightarrow 1$

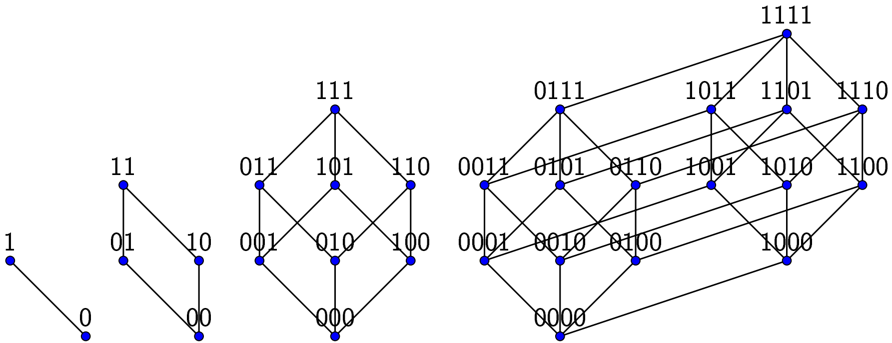

Figure: Hypercubes Q_1, Q_2, Q_3 and Q_4.

Fibonacci cubes

Alphabet $\mathcal{T} = \{0, 1\}$. Adjacency $0 \leftrightarrow 1$

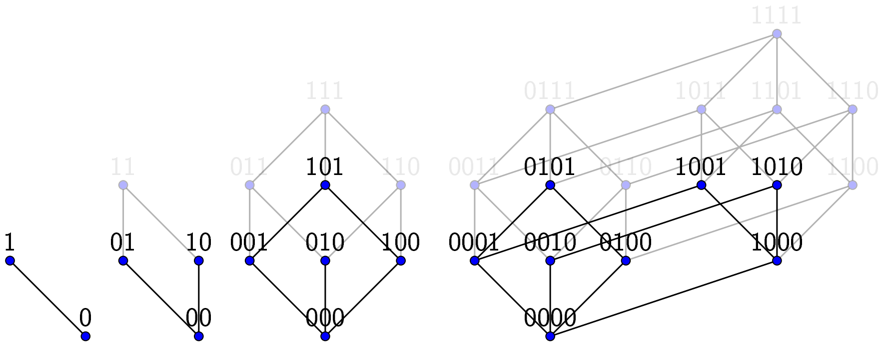

Figure: Fibonacci cubes $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.

Pell graphs[1]

Alphabet $\mathcal{T} = \{0, 1, 2\}$. Adjacency $0 \leftrightarrow 1$ or $11 \leftrightarrow 22$

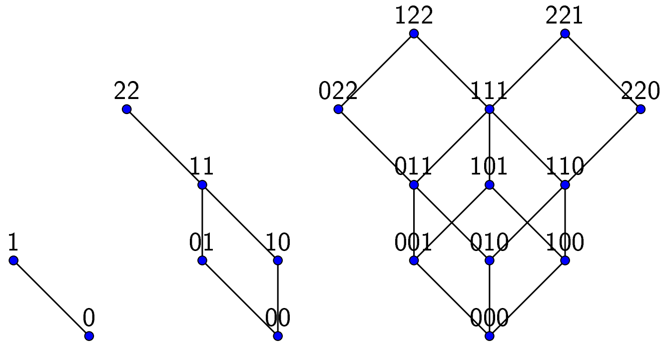

Figure: Pell graphs Π_1, Π_2 and Π_3.

[1] E. Munarini, *Pell graphs*, Discrete Mathematics, **342**,2019

Graphs of linear recurrences of length two

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$V(\Pi_n^a) = \mathcal{S}_n^a$.

For $\alpha = \alpha_1 \cdots \alpha_n$ and $\beta = \beta_1 \cdots \beta_n$ we define

$$\overline{h}(\alpha, \beta) = \sum_{k=1}^{n} |\alpha_k - \beta_k|.$$

Then α and β are adjacent if and only if $\overline{h}(\alpha, \beta) = 1$.

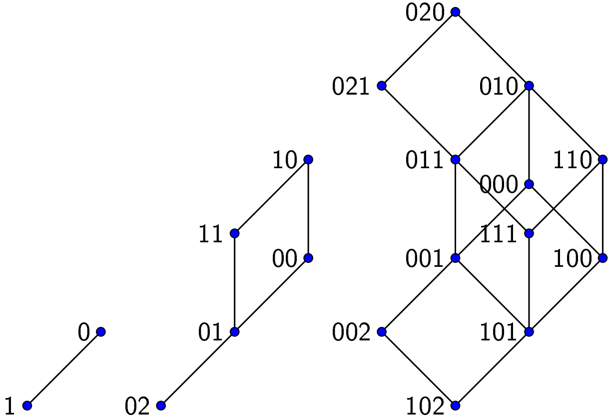

Figure: Graphs Π^2_1, Π^2_2 and Π^2_3.

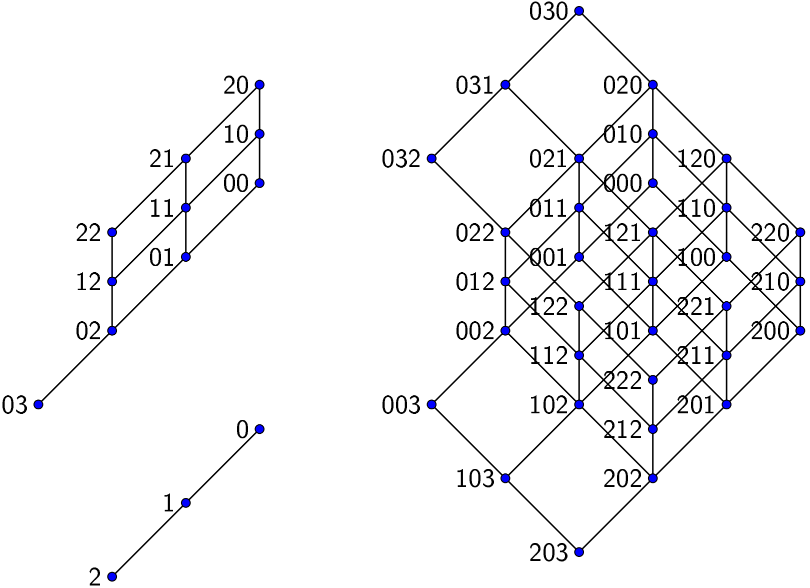

Figure: Graphs Π_1^3, Π_2^3 and Π_3^3.

Decompositions of graph Π_n^a

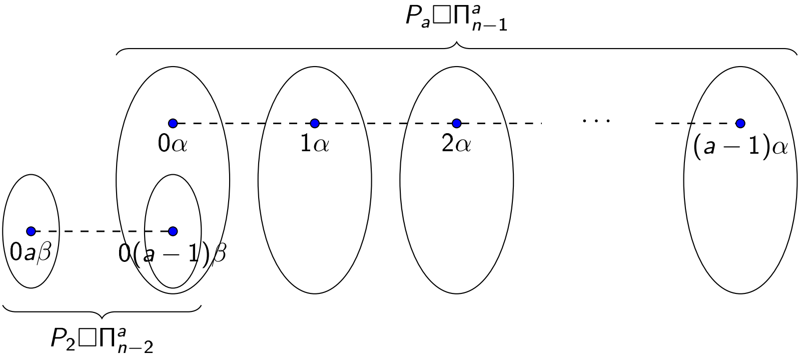

Figure: Canonical decomposition $\Pi^a_n = \Pi^a_{n-1} \oplus \cdots \oplus \Pi^a_{n-1} \oplus \Pi^a_{n-2}$.

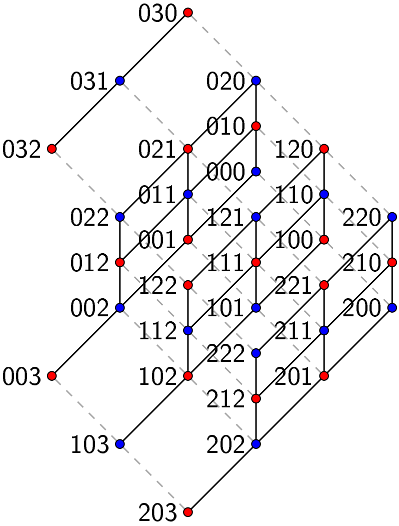

Figure: $\Pi_3^3 = \Pi_2^3 \oplus \Pi_2^3 \oplus \Pi_2^3 \oplus \Pi_1^3$.

Number of edges

$$e_n^a = a \cdot e_{n-1}^a + e_{n-2}^a + s_n^a - s_{n-1}^a.$$

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The number of edges in graph Π_n^a is

$$e_n^a = \sum_{k=0}^{n} (-1)^{n+k} \left\lceil \frac{n+k}{2} \right\rceil \binom{\lfloor \frac{n+k}{2} \rfloor}{k} a^k$$

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Klavžar [2] proved that $|E(\Gamma_n)| = F_{n+1} + \sum_{k=1}^{n-2} F_k F_{n+1-k}$. But since $\Pi_n^1 = \Gamma_{n-1}$, we obtained identity

$$\sum_{k=0}^{n} (-1)^{n+k} \left\lceil \frac{n+k}{2} \right\rceil \binom{\lfloor \frac{n+k}{2} \rfloor}{k} = \sum_{k=0}^{n} F_k F_{n-k}.$$

[2] S. Klavžar, *On median nature and enumerative properties of fibonacci-like cubes*, Discrete Mathematics, **299**,2005

Sequence s_n^a defined with initial values $s_0^a = 1$, $s_1^a = a$ and recursive relation

$$s_n^a = as_{n-1}^a + s_{n-2}^a$$

satisfies well-known identity

$$s_n^a = \sum_{k\geq 0} \binom{n-k}{k} a^{n-2k}.$$

Theorem

Graph Π_n^a can be decomposed into F_{n-1} lattice graphs, where F_n denotes n-th Fibonacci number, i.e.,

$$\Pi_n^a = \bigoplus_{k\geq 0} \binom{n-k}{k} P_a^{n-2k}.$$

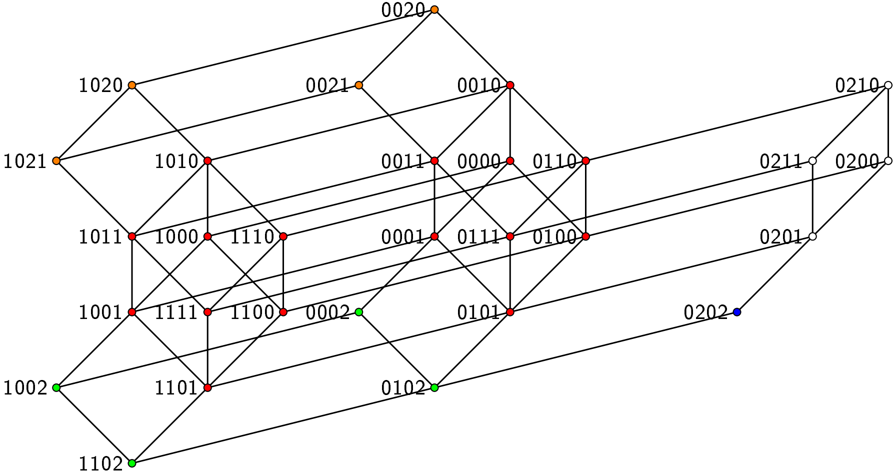

Figure: Decomposition of $\Pi_4^2 = P_2^4 \oplus P_2^2 \oplus P_2^2 \oplus P_2^2 \oplus P_2^0$.

Theorem

Graphs Π_n^1 *and* Γ_{n-1} *are isomorphic.*

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For any $a \geq 2$ *we have embedding* $\Pi_n^a \subseteq \Pi_{(2a-2)n}^1 = \Gamma_{(2a-2)n-1}$.

A median of three vertices is vertex that lies on the shortest path between every two of three vertices. We say that graph is a median if every three vertices have unique median.

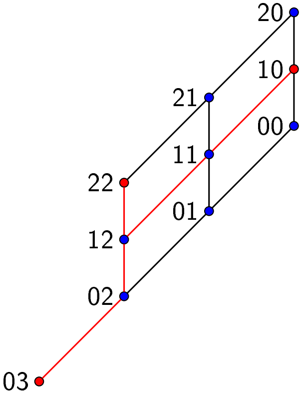

Figure: Median of vertices 10, 22 and 03 is a vertex 12 .

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A graph G is median graph if and only if G is connected induced subgraph of an n-cube such that with any three vertices of G their median in n-cube is also a vertex of G.

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Theorem

For $a \geq 1$ and $n \geq 0$, graph Π_n^a is a median graph.

Radius and diameter of graph Π_n^a.

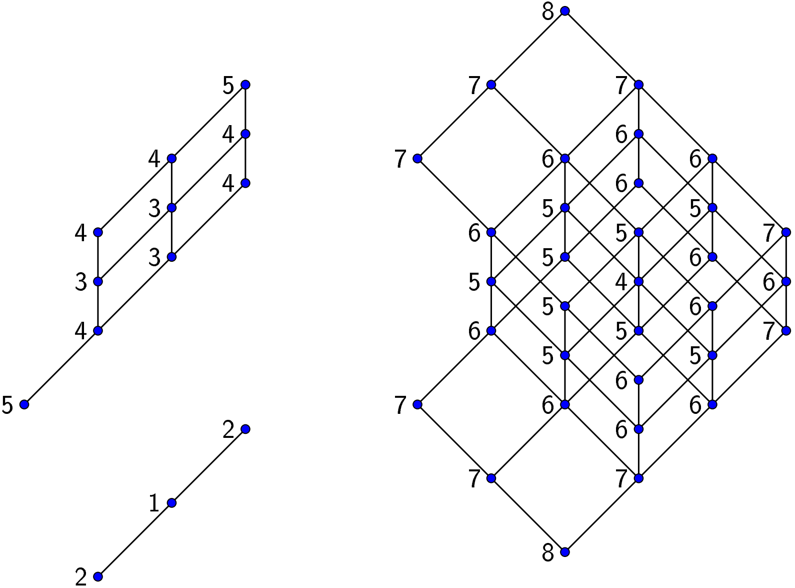

Figure: Graph Π_n^3 for $n = 1, 2, 3$ with eccentricity of every vertex.

By eccentricity $e(v)$ of some vertex $v \in V(G)$ we mean distance from the vertex farthest from it. More precisely, $e(v) = \max_{w \in V(G)} d(v, w)$.

The radius of G, denoted by $r(G)$, is minimum eccentricity of the vertices, and diameter of G, denoted by $d(G)$, is maximum eccentricity.

Subset $Z(G) = \{v \in V(G) : e(v) = r(G)\} \subset V(G)$ is called center of G.

Subset $P(G) = \{v \in V(G) : e(v) = r(G)\} \subset V(G)$ is called periphery of G.

Theorem

For $a \geq 1$ and $n \geq 0$ we have

$$r(\Pi_n^a) = \left\lfloor \frac{a}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{a}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor. \tag{2.1}$$

Theorem

For $a \geq 1$ and $n \geq 0$ we have

$$d(\Pi_n^a) = an - 1. \tag{2.2}$$

Periphery consists of two vertices $0a0a \cdots 0a(0)$ and $(a-1)0a \cdots a0(a)$.

Corollary

For a and n odd, center $Z(\Pi_n^a) = \{\hat{\epsilon}\}$. $|Z(\Pi_n^a)| = 1$

$Z(\Pi_7^5) = \{2222222\}$.

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Let $\epsilon = \left\lfloor \frac{a}{2} \right\rfloor$ and $\hat{\epsilon} = \epsilon \cdots \epsilon \in V(\Pi_n^a)$, as before. We define $U_n^a \subseteq V(\Pi_n^a)$ to be set containing all vertices such that

1. they differ from $\hat{\epsilon}$ in exactly one position, i.e., they have form $\epsilon \cdots \epsilon \alpha \epsilon \cdots \epsilon$, where $\alpha = \epsilon$ or $\alpha = \epsilon \pm 1$,
2. letter $\epsilon + 1$ can only appear on even positions,
3. letter $\epsilon - 1$ can only appear on odd positions.

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Corollary

For a odd and n even, center $Z(\Pi_n^a) = U_n^a \cup \{\hat{\epsilon}\}$. $|Z(\Pi_n^a)| = n + 1$.

$Z(\Pi_6^5) = \{122222, 221222, 222212, 222222, 222223, 222322, 232222\}$.

Let $\epsilon = \frac{a}{2}$ and $V_n^a \subseteq Z(\Pi_n^a)$ be set containing all vertices with letters ϵ or $\epsilon - 1$ such that $\epsilon - 1$ does not appear immediately after ϵ-block of odd length.

Corollary

For a even, center $Z(\Pi_n^a) = V_n^a$. $|Z(\Pi_n^a)| = f_{n+2}$.

$Z(\Pi_4^4) = \{1111, 1112, 1122, 1221, 1222, 2211, 2212, 2222\}$.

Theorem

For $a, n \geq 1$, graphs Π_n^a contain Hamiltonian path.

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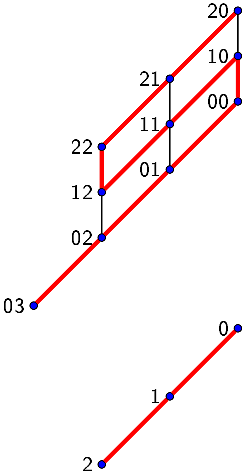

Figure: Hamiltonian path of Π_3^1, Π_3^2 and Π_3^3.

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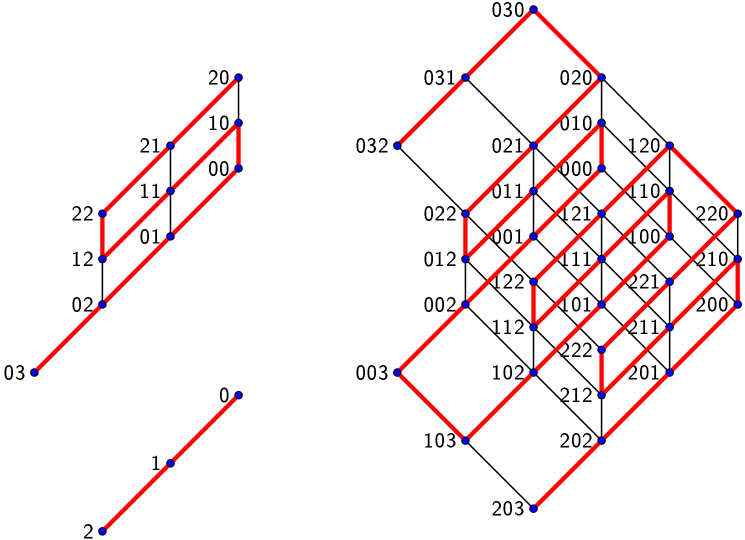

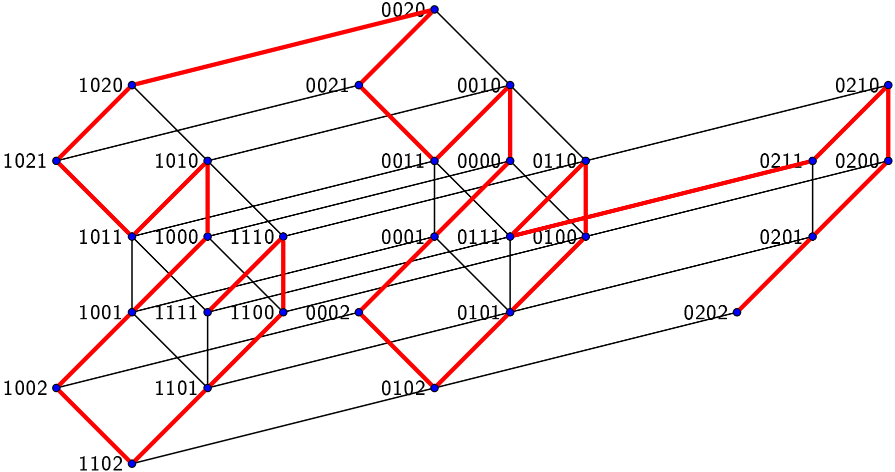

Figure: Hamiltonian path of Π_4^2 with endpoints 1111 and 0202.

For even a: endpoints $0a \cdots 0a(0)$ and $(a-1)\cdots(a-1)$

Thank you for your attention.