Novel constructions of normal covers of the complete bipartite graphs $K_{2^n 2^n}$

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Timeline

1960+
Norman Biggs
Distance-transitive graphs

2001
Cai Heng Li
Basic 2-arc-transitive graphs prime power order

2023+
Dan Hawtin, CP, Jin Xin Zhou constructing covers of 2-power order of complete bipartite graphs

1947+
Bill Tutte

Cubic s-arc-transitive graphs

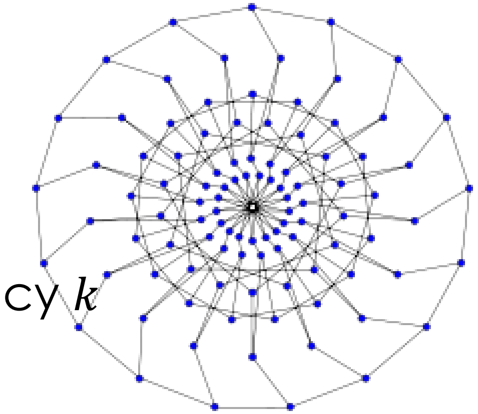
1993
CP
Normal quotients of 2-arc transitive graphs; Ivanov + CP affine (bi)primitive classified

2000-20
Constructions of semi-symmetric graphs from various classes of groups

Background: study families of graphs

$Particularly\ edge - transitive\ graphs$

Biggs—Smith graph
Largest distance transitive graph of valency 3



Distance transitive graphs

$\Gamma = (\Omega, E)$ graph with vertex set Ω, edge set E, and diameter d

For $0 \leq i \leq d$, let $\Gamma_i = \{(\alpha, \beta) | distance(\alpha, \beta) = i\}$

G –distance transitive if G is transitive on each Γ_i (and Γ connected)

From Sims' Conjecture 1983: for each $k \geq 3$, $\exists$ only finitely many DTGs of valency k

Always G transitive on vertex set Ω, on the edge set E, and arc set Γ_1

1971 D.H.Smith - imprimitive DTGs well understood – in terms of links to primitive ones … taking "bipartite halves" and "antipodal quotients"

Background: study families of graphs

$$Particularly\ edge - transitive\ graphs$$

All graphs finite

Distance transitive graphs

$\Gamma = (\Omega, E)$ graph with vertex set Ω, edge set E, and diameter d

For $0 \leq i \leq d$, let $\Gamma_i = \{(\alpha, \beta) | distance(\alpha, \beta) = i\}$

G –distance transitive if G is transitive on each Γ_i

1987 CP, Saxl, Yokoyama: Applied O'Nan-Scott Theorem

To G - DTGs with G primitive on vertices

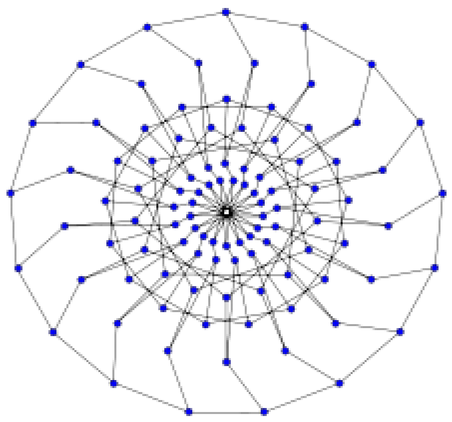
G is affine - Ω finite vector space

G is almost simple

Γ is a Hamming graph or complement

Launched program to classify all finite DTGs

Biggs—Smith graph
Largest distance transitive graph of valency 3



Can similar approaches work for other families of connected graphs?

$Tutte's\ work\ on\ s - arc\ transitive\ graphs$

Connected graph: $\Gamma = (V, E)$ subgroup $G \leq Aut(\Gamma)$; G transitive on V and on 2-arcs

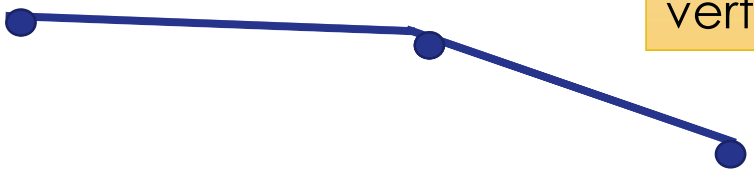
Suitable reduction strategies:

- *Reduced graphs should have same transitivity properties*
- *Tools available to study "basic graphs"*
- *Methods for studying preimages*

2-arc: vertex triple (a, b, c)

G transitive on 2-arcs and on vertices

s-arc: similarly: vertex tuple $(a_1 \dots a_{s+1})$, s edges …



Answer: normal graph quotients

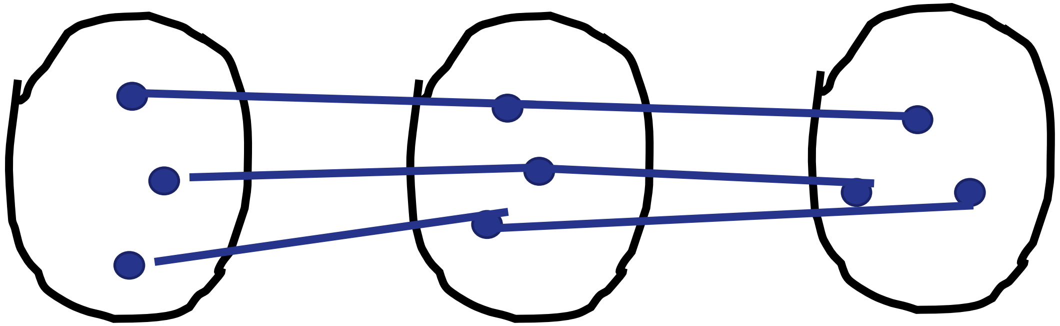
Input: *connected graph* $\Gamma = (V, E)$ and

- $G \leq Aut(\Gamma)$, G transitive on E;
- normal subgroup N of G, usually N intransitive on V

Output: **normal quotient** $\Gamma_N = (V_N, E_N)$ with

- V_N set of N–orbits in V
- E_N N–orbit pairs connected by at least one edge

Γ connected implies that Γ_N is connected



G is transitive on E_N

Answer: normal graph quotients

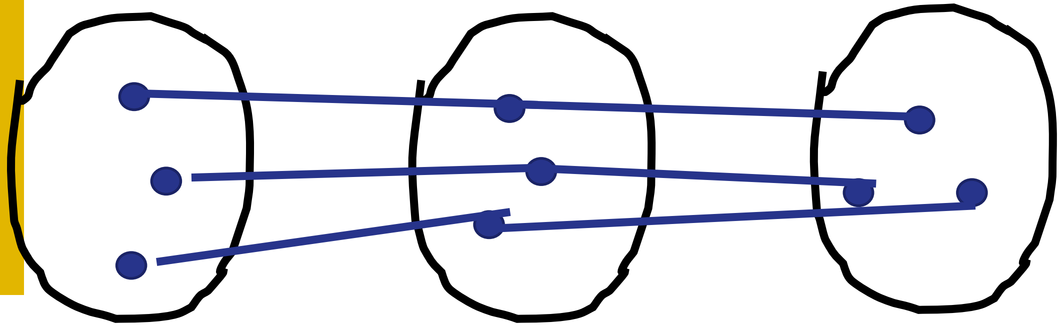
Input: *connected graph* $\Gamma = (V, E)$ and

- $G \leq Aut(\Gamma)$, G transitive on V and on 2-arcs of Γ
- normal subgroup N of G, N with > 2 orbits on V

Output: **normal quotient** $\Gamma_N = (V_N, E_N)$ with

- $G/N \leq Aut(\Gamma_N)$; G/N transitive on V_N and on 2-arcs of Γ_N
- Γ *cover of* Γ_N

Γ a COVER of Γ_N if adjacent parts "joined by" a perfect matching



G/N is transitive on E_N

Basic normal quotients

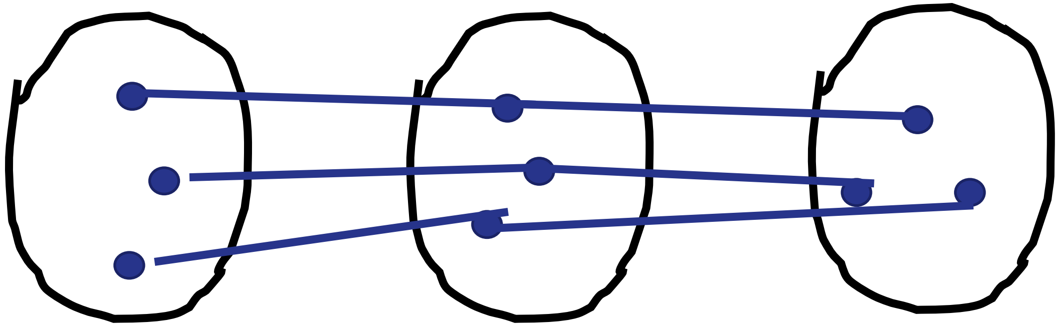
CP 1993

Given: connected graph $\Gamma = (V, E)$ and

- $G \leq Aut(\Gamma)$, G transitive on V and on 2-arcs of Γ

(Γ, G) called *basic* if every normal subgroup $N \neq 1$ of G is either transitive on V, or has two (equal-sized) orbits in V

Outcome: **each** (Γ, G) **has at least one basic normal quotient** $(\Gamma_N, G/N)$ and Γ is a cover of Γ_N



Choose N maximal such that N has at most two orbits in V

Studying 2-arc-transitive graphs

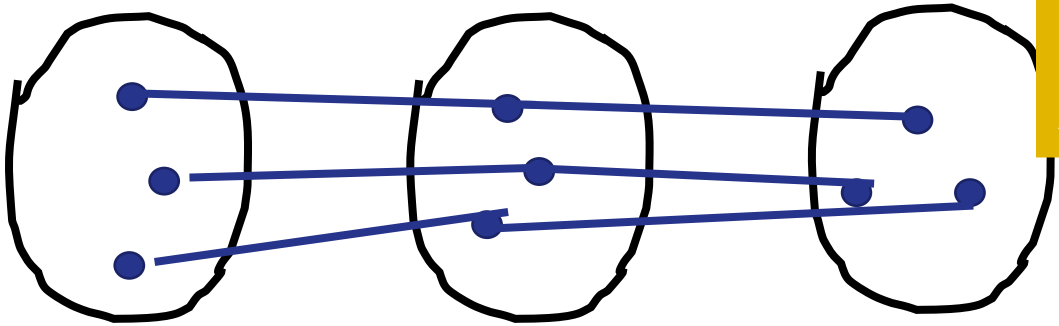
(Γ, G) *basic* if every normal subgroup N of G is either transitive on V, or has two (equal-sized) orbits in V

Challenge 1: **find all basic** (Γ, G) **in some families**

CP 1993

Challenge 2: **find normal covers of a basic** (Γ, G)

Tools: 1. quasiprimitive O'Nan-ScottTheorem; 2. voltage graphs

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Basic 2-arc-transitive p-power order

Challenge 1: (Γ, G) *basic* and $|V| = p^m$ for some prime p.

1993 Ivanov & CP:

- **found all basic (Γ, G) where $\exists$ elementary abelian normal subgroup N of G (regular on V)**

2001 Li: **found all additional basic (Γ, G) of p-power order**

- Which are K_{p^n}, $K_{2^n,2^n}$, **or** $K_{2^n,2^n} - 2^n K_2$ **with non-affine G**

Tools: quasiprimitive O'Nan-Scott Theorem

All 2-arc-transitive p-power order

Problem: **find 2–arc–transitive normal covers of p–power order of these basic** (Γ, G)

2001 Li: **posed this problem after writing:**

"We are inclined to think that non-basic 2-arc transitive graphs of prime-power order would be rare and hard to construct"

Tools: voltage assignments; Malnic et al

This was an important Motivation for "our" investigation – i.e. Dan Hawtin, CP, Jin Xin Zhou


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Normal 2-arc-trans covers of $K_{2^n,2^n}$

2023 Hawtin, CP, Zhou: (Γ, G) connected 2−arc−transitive graph with normal quotient $\Gamma_N = K_{2^n,2^n}$ where N normal in G implies

a) Either Γ is a Cayley graph

b) Or N < H < G with H normal in G, N=H', and H is an **n-dimensional mixed dihedral group**, and the **line graph** of Γ is a Cayley graph for H

Cayley graphs feature in each case

What is an n-dimensional mixed dihedral group?

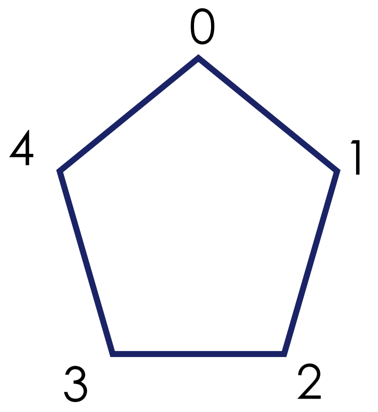
Cayley graphs

$Start\ with\ group$ G $and\ subset\ C$: $Define\ \Gamma = Cay(G; C)$

- to have vertex set G and edges all pairs $\{ g, cg\}$ for $g \in G, c \in C$

We want

- No loops so 1_G not in C
- Undirected graph so C inverse-closed, $c \in C \Rightarrow c^{-1} \in C$
- Connected graph so C generates G



$G = Z_5, C = \{ 1, 4 \}$

Affine symmetries of Cayley graphs

Group G and inverse – closed generating set C: $\Gamma = Cay(G;C)$

- Reminder: edges all pairs $\{g, cg\}$ for $g \in G, c \in C$

"Affine symmetries" $Aff(\Gamma) = G.Aut(G;C) \leq Aut(\Gamma)$

- "Translations" $x: g \to gx$ sends each edge $\{g, cg\}$ to edge $\{gx, cgx\}$ get copy of $G \leq Aut(\Gamma)$ [transitive on vertices]
- "Automorphisms" for $\sigma \in Aut(G)$ with $c^\sigma \in C\ \forall\ c \in C$, $\sigma: g \to g^\sigma$ sends each edge $\{g, cg\}$ to edge $\{g^\sigma, c^\sigma g^\sigma\}$ get subgroup $Aut(G;C) \leq Aut(\Gamma)$ [fixes 1, acts on C]
- $\Gamma = Cay(G;C)$ is **Xu-normal** if $Aff(\Gamma) = Aut(\Gamma)$

Ming Yao Xu

n-dimensional mixed dihedral group

Let H be a finite group and X, Y < H. Then H is an n−dimensional mixed dihedral group relative to X and Y if

- $X \cong Y \cong C_2^n$
- $H = \langle X, Y \rangle$ and
- $H/H' \cong X \times Y$

Examples: which suggested the name

Choose any n even integers $m_1, \ldots, m_n$

- $D_{2m_i} = \langle x_i, y_i \rangle$ with $x_i^2 = y_i^2 = 1$ and $|x_i y_i| = m_i$
- $H = D_{2m_1} \times \cdots \times D_{2m_n} = \langle X, Y \rangle$ with $X = \langle x_1, \ldots, x_n \rangle, Y = \langle y_1, \ldots, y_n \rangle$

n-dimensional mixed dihedral group

Let H be a finite group and X, Y < H. Then H is an n−dimensional mixed dihedral group relative to X and Y if

- $X \cong Y \cong C_2^n$
- $H = \langle X, Y \rangle$ and
- $\mathrm{H}/\mathrm{H}' \cong X \times Y$

In case b) where N < H < G with H normal in G

- N=H' and the line graph of Γ is the Cayley graph
- C(H,X,Y) = Cay(H, S) with S = $(X \cup Y) \setminus \{1\}$

Graphs for n-dim mixed dih'l groups

- $X, Y < H,\ \ X \cong Y \cong C_2^n,\ \ H = \langle X, Y \rangle$ and $\mathrm{H/H'} \cong X \times Y$

The Cayley graph: C(H,X,Y) = Cay(H, S) with S = $(X \cup Y) \setminus \{1\}$

Two families of maximal cliques (complete subgraphs)

- $\{Xh \mid h \in H\}$ and $\{Yh \mid h \in H\}$

These form the vertex set of its clique graph Σ(H,X,Y) with

- $E\Sigma = \{\{Xh, Yg\} \mid Xh \cap Yg \neq \emptyset\}$

Properties of the clique graph

The clique graph Σ(H,X,Y) of C(H,X,Y) has

- $V\Sigma = \{Xh \mid h \in H\} \cup \{Yh \mid h \in H\}$
- $E\Sigma = \{ \{Xh, Yg\} \mid Xh \cap Yg \neq \emptyset \}$

Edges from X: to Yx
Edges from Y: to Xy

$\Sigma = \Sigma(H,X,Y)$ is bipartite with valency 2^n [explain why]

- H has two orbits on $V\Sigma$ and H is transitive on $E\Sigma$
- Σ is a cover of its normal quotient $\Sigma_{H'} = K_{2^n,2^n}$
- $\phi: z \to \{Xz, Yz\}$ defines isomorphism between $\Gamma = C(H,X,Y)$ and the line graph of Σ (bijection from H to $E\Sigma$)

Properties of the clique graph

The clique graph $\Sigma(H,X,Y)$ of C(H,X,Y) has

- $V\Sigma = \{Xh \mid h \in H\} \cup \{Yh \mid h \in H\}$
- $E\Sigma = \{ \{Xh, Yg\} \mid Xh \cap Yg \neq \emptyset \}$

Edges from X: to Yx
Edges from Y: to Xy

Extra properties needed:

- $|V\Sigma| = 2^M$ for some M Holds iff H is a 2-group
- Σ 2-arc-transitive Sufficient condition is that $\mathrm{Aff}(\Gamma)$ is edge-transitive on $\Gamma = \mathrm{C(H,X,Y)}$

First explicit construction of n-dimensional mixed dihedral groups

- Set $X = \langle x_1, \ldots, x_n \rangle$, $Y = \langle y_1, \ldots, y_n \rangle$, and

$H = \langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ satisfying the following defining relations, for $x, x' \in X, y, y' \in Y$

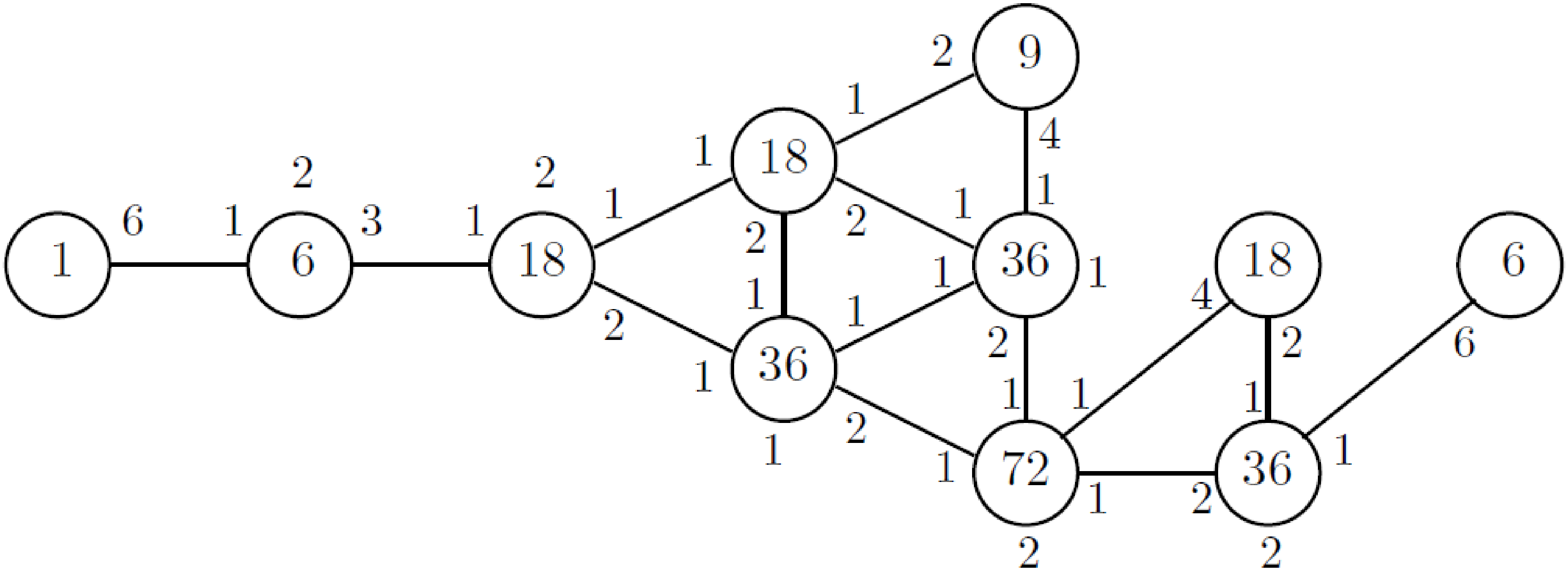
- $x^2 = y^2 = 1, [x, x'] = [y, y'] = 1, [x, y]^2 = 1,$
- $[[x, y], x'] = [[x, y], y'] = 1$

Properties:

- H is an n-dimensional mixed dihedral group; $H/H' \cong X \times Y \cong C_2^{2n}$
- $\mathrm{Aut}(H;S) = \big(\mathrm{GL}(X) \times GL(Y)\big).2$ is transitive and faithful on S
- So $\mathrm{Aff}(\Gamma)$ edge − transitive on $\Gamma = \mathrm{C}(\mathrm{H,X,Y})=\mathrm{Cay}(\mathrm{H, S})$
- Hence Σ(H,X,Y) is a 2-arc transitive normal cover of $K_{2^n,2^n}$

- Also H is a 2-group of order 2^{n^2+2n} and $|V\Sigma| = 2^{n^2+n+1}$
- And C(H,X,Y) is a normal Cayley graph

For the smallest case n=2, here is the distance diagram for the stabiliser in Aut(Γ) for $\Gamma = $ C(H,X,Y)



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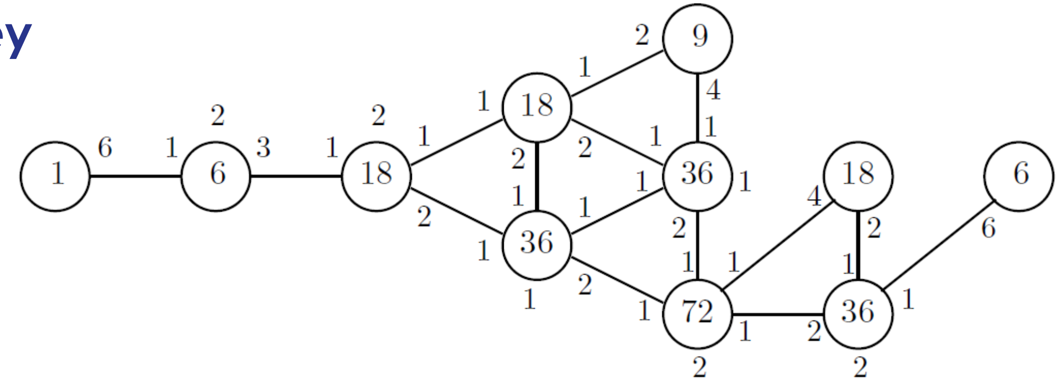
Also aware of another question

Question (Chen, Jin, Li 2019) Is there a **normal Cayley graph** that is 2-distance-transitive, but is neither distance-transitive nor 2-arc-transitive?



All our examples have these properties

Huang, Feng and Zhou (2022) answered with a different construction

Review construction: symmetric in x's and y's nilpotent class 2, …modifications?

H= $\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle$ satisfying the following defining relations, for $i, j, k \leq n$

- $x_i^2 = y_i^2 = 1, [x_i, x_j] = [y_i, y_j] = 1, [x_i, y_j]$^2 $= 1,$
- $[[x_i, y_j], x_k] = [[x_i, y_j], y_k] = 1$

And set

- X=$\langle x_1, \dots, x_n \rangle$, $Y = \langle y_1, \dots, y_n \rangle$

Why would we care?

Semisymmetric graphs Γ

Definition: *Γ is regular & $Aut(\Gamma)$ is transitive on $E\Gamma$ but not on $V\Gamma$*

Consequences: *Γ bipartite and $Aut(\Gamma)$–orbits in $V\Gamma$ two biparts*

Non-examples:

- $K_{a,b}$ *with* $a \neq b$ $Aut(\Gamma)$ *is transitive on* $E\Gamma$ *but not on* $V\Gamma$ *but* Γ *not regular*
- *For bipartite G-arc-transitive Γ is regular and index 2 subgroup* G^+ *is transitive on* $E\Gamma$ *but not on* $V\Gamma$ … *but* … $G^+ \neq Aut(\Gamma)$

Semisymmetric graphs Γ

Examples:

- *1967 Folkman: first constructions - from abelian groups;*
- *Folkman asked what are all possible* $v = |V\Gamma|$ *and* $k =$ *valency of* Γ *?*
- *Ivanov (1987), Conder et al (2006):* $k = 3,\ v \leq 728$
- *Bi-Cayley graphs used to construct examples*
- *Zhou&Feng (2016)* $k = 3,\ v = 2^{2n+7}$
- *Conder et al (2020)* $k \to \infty$

Ming Yao Xu noticed: no examples of 2 − power order

Observation: all the last examples used 2 − generated groups

Review construction: symmetric in x's and y's nilpotent class 2, …modifications?

H= $\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle$ satisfying the following defining relations, for $i, j, k \leq n$

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- $[[x_i, y_j], x_k] = [[x_i, y_j], y_k] = 1$

And set

- X=$\langle x_1, \dots, x_n \rangle$, $Y = \langle y_1, \dots, y_n \rangle$

H needs 2n generators

Second construction:

H= $\langle X_0 \cup Y_0 \mid R \rangle$ where $X_0 = \{x_1, \dots, x_n\}, Y_0 = \{y_1, \dots, y_n\}$ and R is the following set of relations,

where $x, x' \in X_0$, $y, y' \in Y_0$, and $z, z', z''\ z''' \in X_0 \cup Y_0$,

- $z^2 = 1, [x,\ x'] = [y, y'] = 1, [x, y]^2 = 1,\quad [no\ change]$
- $[[y, x], y'] = 1, [[x, y], z]^2 = 1, [[[z, z'], z'',]z'''] = 1$ *[different]*

H has nilpotency class 3, and relations for x and y different

Second construction: facts

2023 Hawtin, CP, Zhou (paper 2):

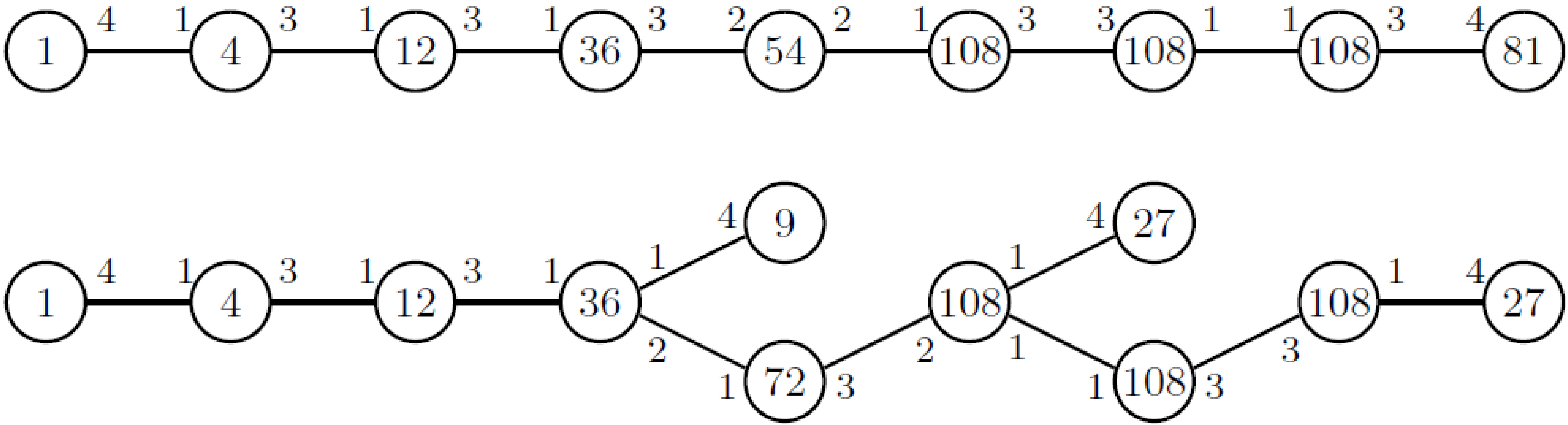
- H= $\langle X_0 \cup Y_0 \mid R \rangle$ is an n−dimensional mixed dihedral 2−group of order $2^{n(n^2+n+4)/2}$ relative to $X = \langle X_0 \rangle$ *and* $Y = \langle Y_0 \rangle$
- Σ(H,X,Y) is semisymmetric and is locally 2-arc transitive

transitive of valency 2^n

H has nilpotency class 3, and relations for x and y different

Second const: smallest example n=2

Distance diagrams from one vertex in each bipart



This smallest one is locally 3-distance transitive: not known in general

Third construction: question of Li et al

2009 Li, Ma, Pan :

Given Γ vertex-transitive, locally primitive, prime power order

- Implies either Γ Cayley graph, or Γ normal cover of $K_{2^n,2^n}$

Question: are ALL the examples Cayley graphs?

2023 Hawtin, CP, Zhou (paper 3): NO!

We give just one example: 4-dim mixed dihedral

Third construction: question of Li et al

2023 Hawtin, CP, Zhou (paper 3):

Our example is Σ(H,X,Y) for a 4-dimensional mixed dihedral group H of order 2^{56}. The graph Σ(H,X,Y) has order 2^{53}.

- Vertex transitive, 2-arc-transitive normal cover of $K_{16,16}$ and not a Cayley graph

Our Questions: are there infinitely many non-Cayley examples?

And what is the smallest such graph?

We give ~ 7 pages of computer assisted analysis in our proof

Some final comments

1: New method of building graph examples from 2-groups – works with appropriate quotients of free groups.

2: Hope this might be relevant for other graph constructions.

Some references:

1. C. E. Praeger, An O'Nan-Scott Theorem for finite quasiprimitive permutation groups, and an application to 2-arc transitive graphs, J. London Math. Soc. (2) 47 (1993), 227-239.

2. A. A. Ivanov, C. E. Praeger, On finite affine 2-arc-transitive graphs, Eur. J. Combin. 14 (1993) 421-444.

3. C. H. Li, Finite s-arc transitive graphs of prime-power order, Bull. London Math. Soc. 33 (2001) 129-137.

4. J. Chen, W. Jin, C.H. Li, On 2-distance-transitive circulants, J. Algebr. Comb. 49 (2019) 179-191.

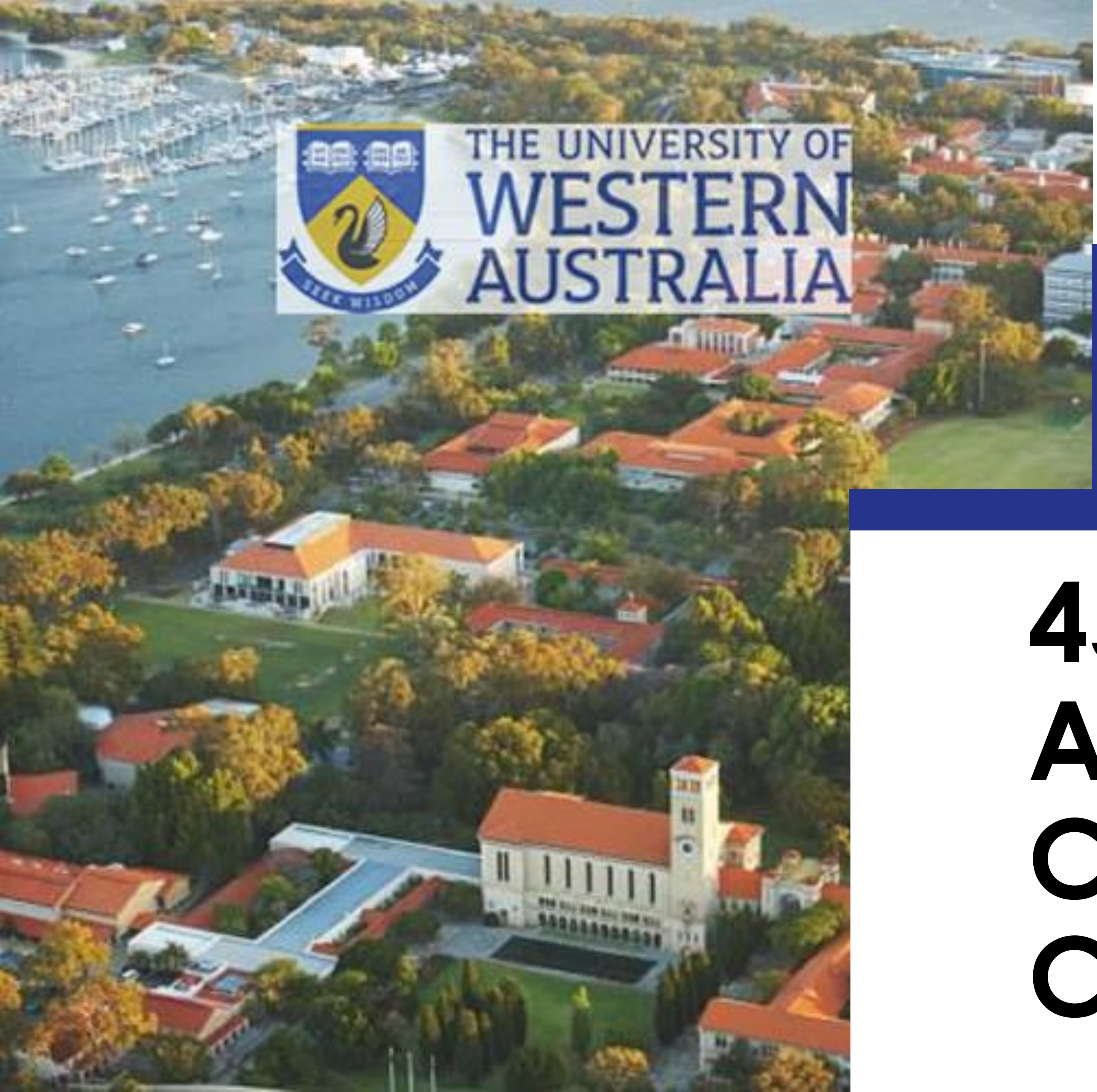
5. C. H. Li, L. Ma, J. Pan, Locally primitive graphs of prime-power order, J. Aust. Math. Soc. 86 (2009) 111–122.

6. D. Hawtin, C. E. Praeger, Jin-Xin Zhou, [all on arXiv]

A characterisation of edge-affine 2-arc-transitive covers of $K_{2^n,2^n}$;

A family of 2-groups and an associated family of semisymmetric, locally 2-arc-transitive graphs

Using mixed dihedral groups to construct normal Cayley graphs, and a new bipartite 2-arc-transitive graph which is not a Cayley graph



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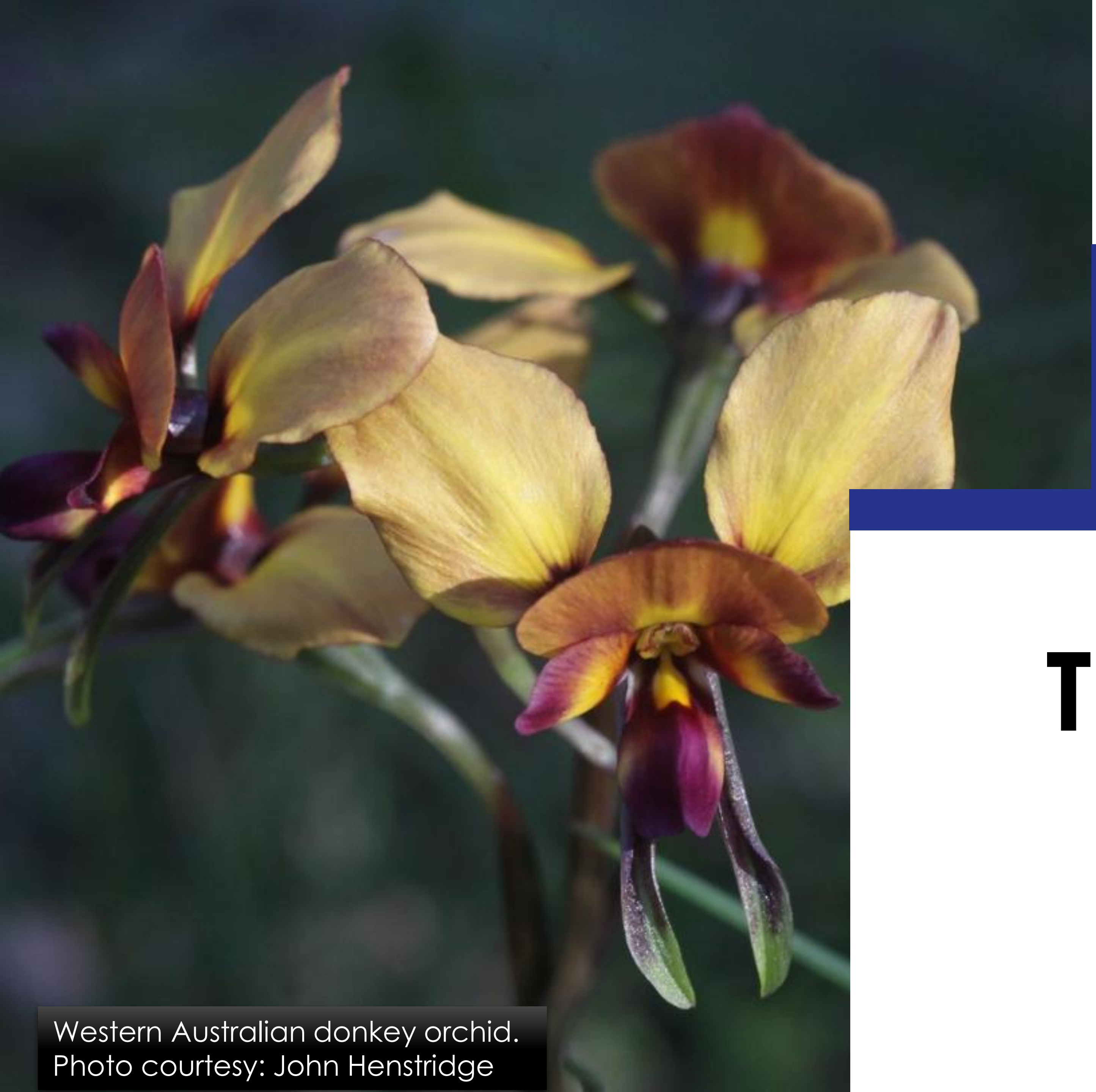

Western Australian donkey orchid.
Photo courtesy: John Henstridge

Thank you all

Stay safe