On two-weight codes invariant under the 3-fold covers of the Mathieu groups M_{22} and $\mathrm{Aut}(\mathrm{M}_{22})$

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Motivation

Some interplay between codes (regarded as modules) and groups

- Given a permutation group G on a finite set Ω and a field $\mathbb{F}$ it is often of considerable interest to know the structure of the permutation module $\mathbb{F}\Omega$ (that is, the vector space over $\mathbb{F}$ with basis Ω considered as an $\mathbb{F}G$ module).
- The G-invariant submodules of $\mathbb{F}\Omega$ can be regarded as linear codes in $\mathbb{F}\Omega$, and one may therefore consider the following:
  - determine the automorphism group of the code
  - determine the weight distribution of the code
  - determine the G-orbit partitions of the code
  - where possible give a geometric (combinatorial) description of the nature of the classes of non-zero codewords of the code, etc.

Definition

Let G be a finite group and $\mathbb{F}$ be a field. The group ring of G over $\mathbb{F}$ is the set of all formal sums of the form

$$\sum_{g \in G} \lambda_g g, \quad \lambda_g \in \mathbb{F}$$

with componentwise addition and multiplication $(\lambda_g)(\mu_h) = (\lambda\mu)(gh)$ (where λ and μ are multiplied in $\mathbb{F}$ and gh is the product in G) extended to sums by means of the distributive law.

- It is a straightforward to verify that the group ring $\mathbb{F}G$ is a vector space over $\mathbb{F}$; and thus we can form $\mathbb{F}G$-modules.
- We now depict the interplay between representations of G and $\mathbb{F}G$-modules.
- In particular, our interest will be in the correspondence between $\mathbb{F}G$-modules and G-invariant subspaces.

Permutation Module

Definition

If Ω is a finite G-set and $\mathbb{F}$ a commutative ring we define $\mathbb{F}G$ to be the free $\mathbb{F}$-module with basis Ω and consider it as a $\mathbb{F}\Omega$-module by extending the action of G on Ω to a $\mathbb{F}$-linear action of $\mathbb{F}G$ on $\mathbb{F}\Omega$. Thus

$$\sum_{g\in G} a_g g \cdot \sum_{w\in G} b_w w = \sum_{g\in G}\sum_{w\in G} a_g b_w (g\cdot w), \quad \text{for } a_g, b_w \in \mathbb{F}.$$

$\mathbb{F}\Omega$ is called the permutation module corresponding to Ω (and $\mathbb{F}$).

The corresponding representation $\delta_\Omega : \mathbb{F}G \longrightarrow End\ \mathbb{F}G$ or its restriction G is called a permutation representation of $\mathbb{F}G$ or G.

Module Structure

Let $G \leq \mathrm{Aut}(C)$

- For $x \in {\mathbb{F}_q}^n$ and a permutation $\sigma \in S_n$ we set

$$\sigma x = \left(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots x_{\sigma^{-1}(n)}\right).$$

- $\mathrm{Aut}(C) = \{\sigma \in S_n \,|\, \sigma x \in C \text{ for all } x \in C\}$
- $C \leq \mathbb{F}_q^n$ as $\mathbb{F}G$-modules
- $\langle \sigma x, \sigma y \rangle = \langle x, y \rangle$, for $x, y \in \mathbb{F}_q^n, \sigma \in G$
- If C is an $\mathbb{F}G$-submodule of $\mathbb{F}\Omega$, then for any $a \in G, \mathbf{c}' \in C^\perp$, and for any $\mathbf{c} \in C$, by the G-invariance of the inner-product we have that

$$\langle a\mathbf{c}', \mathbf{c} \rangle = \langle a\mathbf{c}', aa^{-1}\mathbf{c} \rangle = \langle \mathbf{c}', a^{-1}\mathbf{c} \rangle = 0,$$

  so $a\mathbf{c}' \in C^\perp$, i.e., $C^\perp$ is G-invariant.
- $C^\perp$ is also an $\mathbb{F}G$-submodule.
- $\mathrm{Aut}(C) = \mathrm{Aut}(C^\perp)$

- $C^* = \mathrm{Hom}_{\mathbb{F}}(C, \mathbb{F})$ becomes a $\mathbb{F}G$-module via $\sigma(f)(c) = f(\sigma^{-1}(c))$
- $\mathbb{F}_q^n / C^\perp \cong C^*$ as $\mathbb{F}G$-modules
- G acts on C and thus $G \leq \mathrm{Aut}(C)$ so that the code C becomes a $\mathbb{F}G$-submodule of the permutation module $\mathbb{F}\Omega$.
- The submodules of $\mathbb{F}\Omega$ can be considered as linear codes with ambient space $\mathbb{F}\Omega$ and ambient basis Ω whereas G acts as a group of automorphisms of any such codes.
- Strategy: Let G be a finite group and Ω a finite G-set. Then the $\mathbb{F}_q G$-submodules of $\mathbb{F}_q\Omega$ are precisely the G-invariant codes (i.e., G-invariant subspaces of $\mathbb{F}_q\Omega$).
- If C is a code of length n, the $(n+1)$-tuple $(w_i(C))_{0 \leq i \leq n}$ where $w_i(C) = |W_i(C)|$ is called the weight distribution of the code C.
- The polynomial $W_C(x) = \sum_{i=0}^{n} A_i x^i$ of degree n where A_i denotes the number of codewords of weight i in C is called the weight enumerator of C. The weight enumerator of a code C and its dual $C^\perp$ are related via MacWilliams' identity.

A unifying approach

In most instances, particularly when the degree of the permutation representation (or the dimension of the permutation module) is sufficiently large, the strategy given above is not very useful.

In

W. Knapp and B. G. Rodrigues.,
A useful tool for constructing linear codes.
J. Algebra **585** (2021), 422–446

we determined a unifying setting in which many binary linear codes can be studied.

- This talk is about codes obtained from permutation modules induced by the action of finite groups.
- ... irreducible modules invariant finite quasisimple almost simple groups

Theorem 1: Let G be a finite group and let V be an $\mathbb{F}G$-module over a finite field $\mathbb{F}$ and let Ω be a G-invariant subset of V. Let $\mathbb{F}\Omega$ be the (formal) permutation module with basis $\overline{\Omega} = \{\overline{\alpha} \,|\, \alpha \in \Omega\}$, where $\overline{\alpha} = (\delta_{\beta,\alpha})_{\beta\in\Omega}$, and $\delta_{\beta\alpha}$ denotes the Kronecker δ function.
Then

$$\rho : \sum_{\alpha\in\Omega} r_\alpha \overline{\alpha} \mapsto \sum_{\alpha\in\Omega} r_\alpha \alpha$$

is an $\mathbb{F}G$-homomorphism of $\mathbb{F}\Omega$ into V with $M = \ker(\rho) = \{\sum_{\alpha\in\Omega} r_\alpha\overline{\alpha} \,|\, \sum_{\alpha\in\Omega} r_\alpha\alpha = 0 \text{ in } V\}$ and image U, where U is the submodule of V generated by Ω. Hence, we have

$$\mathbb{F}\Omega/M \cong U \quad \textit{and} \quad M^\perp \cong U^*,$$

where $M^\perp$ denotes the submodule of $\mathbb{F}\Omega$ orthogonal to M with respect to the canonical bilinear form on $\mathbb{F}\Omega$ and $U^* = \mathrm{Hom}(U, \mathbb{F})$ denotes the $\mathbb{F}G$-module dual to U in the sense of representation theory.

Proof: The action of G on Ω is given by restricting the action of $G(\subseteq \mathbb{F}G)$ on V.
The theorem is basically just a restatement of the universal property of the permutation module as a free structure over Ω using in addition some elementary facts of representation theory and linear algebra.
Now, fill out the remaining details of the proof. ■

The submodule $E = M^{\perp} = (\mathrm{Ker}(\tau))^{\perp}$ in Theorem 1 is called the evolution of V along Ω as an $\mathbb{F}G$-module.

We study linear two-weight codes invariant under the triple cover of the Mathieu groups M_{22} and $\mathrm{Aut}(\mathrm{M}_{22})$.

- A code C is a **two-weight code** if all its nonzero codewords have weight w_1 or w_2 $(w_1 < w_2)$, for some w_1, w_2.
- C is called **projective** if no two columns of a generator matrix are linearly dependent, i.e. the columns represent pairwise distinct points in a projective $(k-1)$-dimensional space or equivalently, if $d(C^{\perp}) \geq 3$.
- If the Hamming weight $\mathrm{wt}(c)$ of every codeword $c \in C$ is divisible by an integer $\Delta > 1$ then C is said to be **Δ-divisible**.

An application of Theorem 1

Proposition 2: Let $G \cong \hat{3}\mathrm{M}_{22}$ then the following hold:
(i) $\mathbb{F}_4$ is a splitting field for G.
(ii) G has a unique absolutely irreducible representation of dimension 6 over $\mathbb{F}_4$. This representation gives a maximal embedding of $\hat{3}\mathrm{M}_{22}$ into $\hat{3}\mathrm{U}_6(2)$.
(iii) Let Ω_i denote the orbits of the action of G on the set of 1-spaces of $V \cong \mathbb{F}_4^6$. Then G has exactly 3 orbits Ω_i of lengths $|\Omega_i| = 693, 1386$ and 2016, for $1 \leq i \leq 3$, with stabilizers subgroups isomorphic to $2^4{:}S_5$, $2^4{:}\mathrm{A}_5$ and $\mathrm{L}_2(11)$, respectively.

Proposition 3: The conditions of Theorem 1 and Proposition 2 are satisfied for $G = \hat{3}\mathrm{M}_{22}$, $V = \mathbb{F}_4^6$ and Ω_i the G-orbit of length $|\Omega_i|$. The Evolution E of V along Ω is a 6-dimensional submodule of the permutation module of length $|\Omega_i| = 693, 1386$ and 2016, for $1 \leq i \leq 3$.

Quaternary two-weight codes invariant under $\hat{3}\mathrm{M}_{22}$

The objects of $V \setminus \{0\}$ consists of 693 elements of Ω_1 which are the isotropic 1-spaces of V, the 1386 elements of Ω_2 represent the type $2B$ vectors, while the 2016 elements of Ω_3 represent the set of non-isotropic 1-spaces in V, i.e, the type $4A$ vectors.

- Using some properties of $V \cong \mathbb{F}_4^6$ in

B G Rodrigues.

Some two-weight codes invariant under the 3-fold covers of the Mathieu groups M_{22} *and* $\mathrm{Aut}(\mathrm{M}_{22})$

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as a direct application of Theorem 1 we examined quaternary two-weight codes of small dimension related with the triple cover $\hat{3}\mathrm{M}_{22}$ of M_{22}

Some quaternary two-weight codes invariant under $\hat{3}\mathrm{M}_{22}$

Proposition 4 Let G be the triple cover $\hat{3}\mathrm{M}_{22}$ of the Mathieu group M_{22}. Let Ω_i be an orbit of length $|\Omega_i|$ for $1 \leq i \leq 3$ of G on the 1-spaces of V as described in Proposition 3.
Let $\mathrm{C}_4(\Omega_i)$ denote the evolution code of V along Ω_i. Then the following hold:
(a) $\mathrm{C}_4(\Omega_1) = [693, 6, 480]_4$ is a two-weight code with weight enumerator $x^0 + 693x^{480} + 3402x^{528}$ and $\mathrm{Aut}(\mathrm{C}_4(\Omega_1)) \cong \hat{3}\mathrm{M}_{22}$;
(b) $\mathrm{C}_4(\Omega_2) = [1386, 6, 1008]_4$ is a two-weight code with weight enumerator $x^0 + 1386x^{1008} + 2709x^{1056}$ and $\mathrm{Aut}(\mathrm{C}_4(\Omega_2)) \cong \hat{3}\mathrm{M}_{22}$;
(c) $\mathrm{C}_4(\Omega_3) = [2016, 6, 1488]_4$ is a two-weight code with weight enumerator $x^0 + 2016x^{1488} + 2079x^{1536}$ and $\mathrm{Aut}(\mathrm{C}_4(\Omega_3)) \cong 3\mathrm{U}_6(2){:}3$.
(d) $\mathrm{C}_4(\Omega_i)$ is a self-orthogonal 4^2-divisible two-weight code for $1 \leq i \leq 3$.

Binary projective two-weight codes invariant under $\hat{3}\mathrm{M}_{22}$:2

For $G \cong \hat{3}\mathrm{M}_{22}$:2 the following hold:

(i) G has a unique absolutely irreducible representation of dimension 12 over $\mathbb{F}_2$.

(ii) Let Λ_i, for $1 \leq i \leq 3$ denote the orbits of the action of G on the set of 1-spaces of $V \cong \mathbb{F}_2^{12}$.

(iii) G has exactly 3 orbits on the set of nonzero vectors in V with stabilizers isomorphic to $2^5{:}S_5$, $2^4{:}(\mathrm{A}_5 \times 2)$, and $\mathrm{L}_2(11){:}2$, the orbit lengths being 693, 1386 and 2016, respectively.

(iv) The orbits Λ_1, Λ_2 consist of singular vectors, while Λ_3 consists of non-singular vectors of V.

Proposition 5 Let G be the triple cover $\hat{3}\mathrm{M}_{22}{:}2$ of the Mathieu group $\mathrm{M}_{22}{:}2$.
Let Λ_i be an orbit of length $|\Lambda_i|$, for $1 \leq i \leq 3$ of G on the nonzero vectors of V.
Let $\mathrm{C}_2(\Lambda_i)$ denote the evolution code of V along Λ_i. Then the following hold:
(a) $\mathrm{C}_2(\Lambda_1) = [693, 12, 320]_2$ is a two-weight code with weight enumerator $x^0 + 693x^{320} + 3402x^{352}$ and $\mathrm{Aut}(\mathrm{C}_2(\Lambda_1)) \cong \hat{3}\mathrm{M}_{22}{:}2$;
(b) $\mathrm{C}_2(\Lambda_2) = [1386, 12, 672]_2$ is a two-weight code with weight enumerator $x^0 + 1386x^{672} + 2709x^{704}$ and $\mathrm{Aut}(\mathrm{C}_2(\Lambda_2)) \cong \hat{3}\mathrm{M}_{22}{:}2$.
(c) $\mathrm{C}_2(\Lambda_3) = [2016, 12, 992]_2$ is a two-weight code with weight enumerator $x^0 + 2016x^{992} + 2079x^{1024}$; and $\mathrm{Aut}(\mathrm{C}_2(\Lambda_3)) \cong \mathrm{O}^+_{12}(2){:}2 = \mathrm{PSO}^+_{12}(2)$.
(d) $\mathrm{C}_2(\Lambda_i)$ is a self-orthogonal 2^5-divisible projective two-weight code.

Observations

- That $\mathrm{Aut}(\mathrm{C}_2(\Lambda_3)) \cong \mathrm{O}_{12}^+(2){:}2 = \mathrm{PSO}_{12}^+(2)$ in part (iii) follows by observing that the full $2A$-centralizer in J_4 is $2_+^{1+12}.\hat{3}\mathrm{M}_{22}{:}2$.
- The action of $\hat{3}\mathrm{M}_{22}{:}2$ on 2_+^{1+12} is given by the embedding of $\hat{3}\mathrm{M}_{22}$ in $\mathrm{SU}_6(2) < \mathrm{O}_{12}^+(2){:}2 = \mathrm{Out}(2_+^{1+12})$.
- Observe that $\mathrm{C}_2(\Lambda_i)$ for $1 \leq i \leq 3$ does not contain the all-ones vector $\mathbf{j}$.
- We construct binary self-complementary linear codes of dimension 13, denoted $\widehat{\mathrm{C}_2}(\Lambda_i)$ which result by adjoining the all-ones vector to $\mathrm{C}_4(\Omega_i)$.
- In fact, $\widehat{\mathrm{C}_2}(\Lambda_i) \setminus \mathrm{C}_2(\Lambda_i)$ consists of the codewords complementary to those of $\mathrm{C}_2(\Lambda_i)$.

Question: what are the combinatorial properties of the code $\widehat{\mathrm{C}_2}(\Lambda_i)$?

Grey-Rankin bound

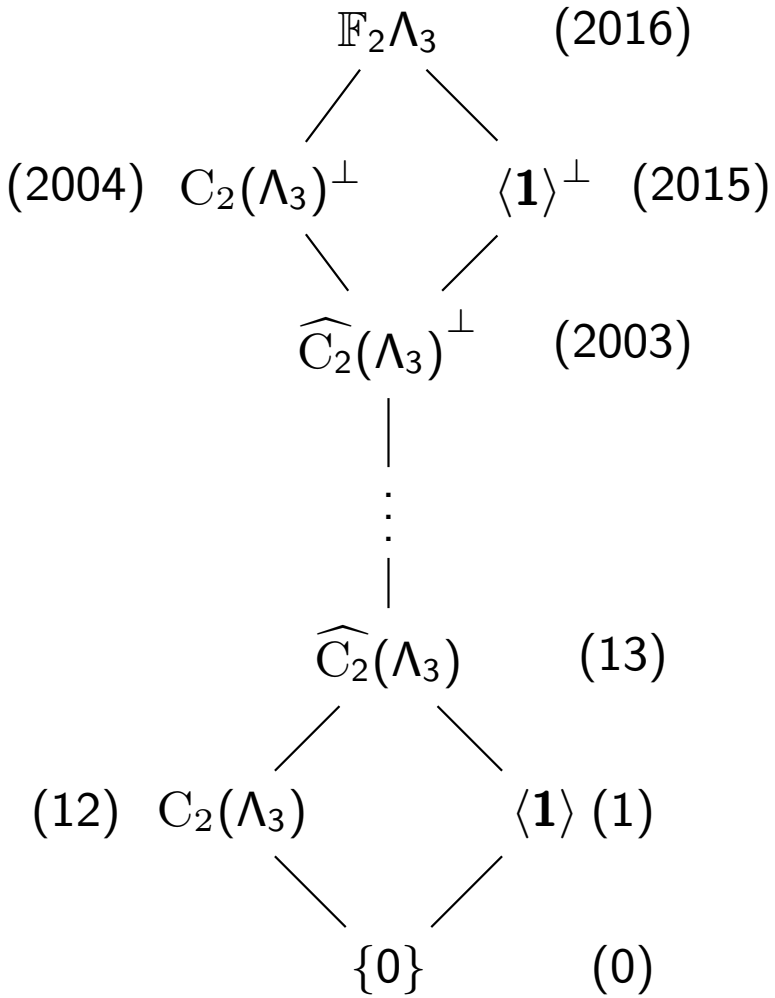
For any length n and minimum distance d, the Grey–Rankin bound is an upper bound for the cardinality of a binary self-complementary code C. It states that

$$|C| \leq \frac{8d(n-d}{n-(n-2d)^2}$$

provided that the right-hand side is positive.

Proposition 6 The following hold for $\widehat{\mathrm{C}_2}(\Lambda_i)$:
(i) $\widehat{\mathrm{C}_2}(\Lambda_i) = \langle \mathrm{C}_2(\Lambda_i), \mathbf{j} \rangle$;
(ii) If $d(\widehat{\mathrm{C}_2}(\Lambda_i)^{\perp})$ denotes the minimum distance of $\widehat{\mathrm{C}_2}(\Lambda_i)^{\perp}$ for each i. Then
$d(\widehat{\mathrm{C}_2}(\Lambda_i)^{\perp}) = 4$.
(iii) $\widehat{\mathrm{C}_2}(\Lambda_1)$ is a projective $[693, 13, 320]_2$ code with weight enumerator $1 + 693x^{320} + 3402x^{341} + 3402x^{352} + 693x^{373} + x^{693}$, and $\mathrm{Aut}(\widehat{\mathrm{C}_2}(\Lambda_1)) \cong \hat{3}\mathrm{M}_{22}{:}2$;
(iv) $\widehat{\mathrm{C}_2}(\Lambda_2)$ is a self-orthogonal, projective 2-divisible $[1386, 13, 672]_2$ code with weight enumerator $1 + 1386x^{672} + 2709x^{682} + 2709x^{704} + 1386x^{714} + x^{1386}$, and $\mathrm{Aut}(\widehat{\mathrm{C}_2}(\Lambda_2)) \cong \hat{3}\mathrm{M}_{22}{:}2$;
(v) $\widehat{\mathrm{C}_2}(\Lambda_3)$ is a self-orthogonal, projective 2^5-divisible $[2016, 13, 992]_2$ code with weight enumerator $1 + 4095x^{992} + 4095x^{1024} + x^{2016}$, and $\mathrm{Aut}(\widehat{\mathrm{C}_2}(\Lambda_3)) \cong \mathrm{PSp}_{12}(2)$. Moreover, $\widehat{\mathrm{C}_2}(\Lambda_3)$ is a binary self-complementary code that meets the Grey-Rankin bound with equality.

Figure: Partial submodule lattice for $\mathbb{F}_2\Lambda_3$ of dimension 2016



SRG from $\widehat{\mathrm{C}_2}(\Lambda_i)$

Based on results of

R. Calderbank and W. M. Kantor.,
The geometry of two-weight codes.
Bull. London Math. Soc. **18** (1986), 97–122,

we construct SRG $\mathcal{L}(\mathrm{C}_2(\Lambda_i))$ in which the vertices are identified with the codewords and two vertices corresponding to codewords x and y are adjacent if and only if $d(x, y) = w_1$
The parameters of $\mathcal{L}(\mathrm{C}_2(\Lambda_i))$ are given by:

$$\begin{aligned} N &= q^k, \\ K &= n(q-1), \\ \lambda &= K^2 + 3K - q(w_1+w_2) - Kq(w_1+w_2) + q^2 w_1 w_2, \\ \mu &= K^2 + K - Kq(w_1+w_2) + q^2 w_1 w_2. \end{aligned} \tag{1}$$

Theorem

Let $\mathcal{L}_i$ denote the graph associated to $\mathrm{C}_2(\Lambda_i)$, for $1 \leq i \leq 3$. Then the parameters of $\mathcal{L}_i$ (respectively of the complementary graph $\overline{\mathcal{L}_i}$) and the structure of the automorphism group are as given in following table:

$\mathcal{L}_i$	N	K	λ	μ	r^f	s^g	$\mathrm{Aut}(\mathcal{L}_i)$
1	4096	693	152	110	$(53)^{693}$	$(-11)^{3402}$	$2^{12}{:}\hat{3}\mathrm{M}_{22}{:}2$
		3402	2818	2862	$(10)^{3402}$	$(-54)^{693}$	
2	4096	1386	482	462	$(42)^{1386}$	$(-22)^{2709}$	$2^{12}{:}\hat{3}\mathrm{M}_{22}{:}2$
		2709	1784	1806	$(21)^{2709}$	$(-43)^{1386}$	
3	4096	2016	992	992	$(32)^{2016}$	$(-32)^{2079}$	$2^{12}{:}\mathrm{PSO}_{12}^{+}(2)$
		2079	1054	1056	$(31)^{2079}$	$(-33)^{2016}$	

Table: *Strongly regular graphs on 4096 vertices related to $\mathrm{C}_2(\Lambda_i)$.*

Some pertinent observations

- $\mathrm{Aut}(\mathcal{L}_3) \cong 2^{12}{:}\mathrm{PSO}^+_{12}(2)$ with vertex stabilizer isomorphic to $\mathrm{PSO}^+_{12}(2)$ of degree 4096 with subdegrees: 1, 2016, 2079 respectively.
- Under the action of $\mathrm{Aut}(\mathcal{L}_3)$ the orbits of lengths 693 and 1386 fuse, giving rise to the orbit of length 2079, and thus a rank 3 action of degree 4096 with subdegrees 1, 2016 and 2079.
- The parameters of $\mathcal{L}_3$ and the structure of the automorphism group can also be deduced from "M. Liebeck. The affine permutation groups of rank 3. Proc. Lond. Math. Soc., **III** Ser. **54** (1987), 477 - 516)", where it is shown that $\mathrm{Aut}(\mathcal{L}_3)$ is a rank 3 affine permutation group.

Tying loose ends

- Taking for blocks the images of the elements of an orbit of length 2016 (or the images of the union of the orbits of lengths 1, 693, and 1386, respectively) under the action of any of the automorphism groups of the graphs given in the above Table, we obtain a 2-$(4096, 2016, 992)$ (respectively a 2-$(4096, 2080, 1024)$) symmetric design with automorphism group isomorphic to $2^{12}{:}\mathrm{PSp}_{12}(2)$.
- Designs with parameters 2-$(4096, 2016, 992)$ are known to satisfy the symmetric difference property (SDP), i.e. designs for which the symmetric difference of any three blocks is either a block or the complement of a block.

Tying loose ends ... continued

Proposition: Let $\mathcal{D}$ be the design formed by taking as blocks the images under $\mathrm{Aut}(\mathcal{L}_i)$, $1 \leq i \leq 3$ of an orbit of length 2016. Then $\mathcal{D}$ is a symmetric 2-$(4096, 2016, 992)$ design with the symmetric difference property and $\mathrm{Aut}(\mathcal{D}) \cong 2^{12}{:}\mathrm{PSp}_{12}(2)$.
Let C be the binary code of $\mathcal{D}$. Then C is a $[4096, 14, 2016]_2$ code and $C^{\perp}$ is a projective $[4096, 4082, 4]_2$ code. Moreover, $\mathrm{Aut}(C) \cong 2^{12}{:}\mathrm{PSp}_{12}(2)$. The weight distribution of C is given as follows:

$$A_0 = A_{4096} = 1, \ A_{2016} = A_{2080} = 4096, \ A_{2048} = 8190.$$

The codewords of weight 2048 span a two-weight subcode C' of C of codimension 1 with parameters $[4096, 13, 2048]_2$ and weight distribution $A_0 = A_{4096} = 1, \ A_{2048} = 8190$. Further, $\mathrm{Aut}(C') \cong 2^{12}{:}\mathrm{PSL}_{12}(2)$.

Still tying loose ends ...

Corollary: The derived and residual designs of $\mathcal{D}$ with respect to a block are a 2-$(2016, 992, 991)$ and a 2-$(2080, 1024, 992)$ design.
The binary codes of these designs have parameters $[2016, 13, 992]_2$ and $[2080, 13, 1024]_2$, respectively with automorphism groups isomorphic to $\mathrm{PSp}_{12}(2)$.
Moreover, these codes meet the Grey-Rankin bound with equality.

Final note:

- According to "D. Jungnickel and V. D. Tonchev. Exponential number of quasi-symmetric SDP designs and codes meeting the Grey-Rankin bound. Des. Codes and Cryptogr., **1** (1991), 247–253", the number of non-isomorphic quasi-symmetric SDP designs grows exponentially.

Finally, tying loose ends ... ended!!

- The exact number of non-isomorphic quasi-symmetric SDP designs with parameters those given in the Corollary, depends on the number of inequivalent univariate bent functions from $\mathbb{F}_{2^m}$ to $\mathbb{F}_2$.
- The number of inequivalent bent functions for m even and up to 10 has been calculated in "A. E. Brouwer and H. van Maldeghem. Strongly regular graphs. Cambridge: Cambridge University Press, 2022, see pp. 196".
- In particular, for $m = 6$ there are exactly 896 inequivalent univariate bent functions.
- So, there are precisely 896 non-isomorphic designs with parameters those those given in the Corollary.

Thank you for your presence !!!!