An Erdős-Ko-Rado theorem for transitive groups of degree a product of two odd primes

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$_ Joint work with

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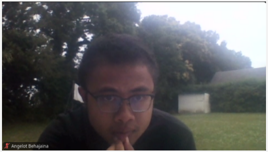

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Pablo Spiga

Karen Meagher

1. The EKR Theorem

2. EKR for transitive groups

3. Derangement graphs

4. Transitive groups with fixed degree

5. The primitive case

6. The imprimitive case

#_ The EKR Theorem

$_ The Erdős-Ko-Rado theorem

Theorem (Erdős-Ko-Rado, 1961)

For any two positive integers such that $n \geqslant 2k+1$, if $\mathcal{F}$ is a collection of k-subsets of $[n] := \{1, 2, \dots, n\}$ such that $|A \cap B| \geqslant 1$, for all $A, B \in \mathcal{F}$, then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1}.$$

Moreover, equality holds if and only if $\mathcal{F}$ is of the form

$$\mathcal{F}_a = \{A \subset [n] : |A| = k \text{ and } a \in A\},$$

for some $a \in [n]$.

#_ EKR for transitive groups

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- The intersection density of the transitive group $G \leqslant \mathrm{Sym}(\Omega)$ is the rational number

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- » $\rho(G) \geqslant 1$.
- » G has the EKR property if $\rho(G) = 1$.
- » G has the strict EKR property if whenever $\mathcal{F} \subset G$ is an intersecting set such that $\rho(G) = \frac{|\mathcal{F}|}{|G_\omega|}$, then $\mathcal{F}$ is a coset of a stabilizer of a point.

$_ Examples

Results

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Theorem (Meagher-Spiga-Tiep, 2015)

If G is a finite 2-transitive group, then G has the EKR property.

$_ A non-example

* Consider $H = \langle (1\ 2)(3\ 4), (3\ 4)(5\ 6) \rangle$ and $g = (1\ 3\ 5)(2\ 4\ 6)$.
* The permutation g is in the normalizer of H in $\mathrm{Sym}(6)$ and $G = \langle H, g \rangle \cong \mathrm{Alt}(4)$ is transitive.
* H is intersecting and $\rho(G) \geqslant \frac{4}{2}$.

#_ Derangement graphs

$_ Derangement graph

If $G \leqslant \mathrm{Sym}(\Omega)$ is transitive and $\mathrm{Der}(G)$ is the set of all derangements of G, then the derangement graph Γ_G of G is the Cayley graph $\mathrm{Cay}(G, \mathrm{Der}(G))$. That is, Γ_G is the graph with

- vertex set G,
- $g \sim_{\Gamma_G} h \Leftrightarrow hg^{-1} \in \mathrm{Der}(G)$.

$_ Example of a derangement graph

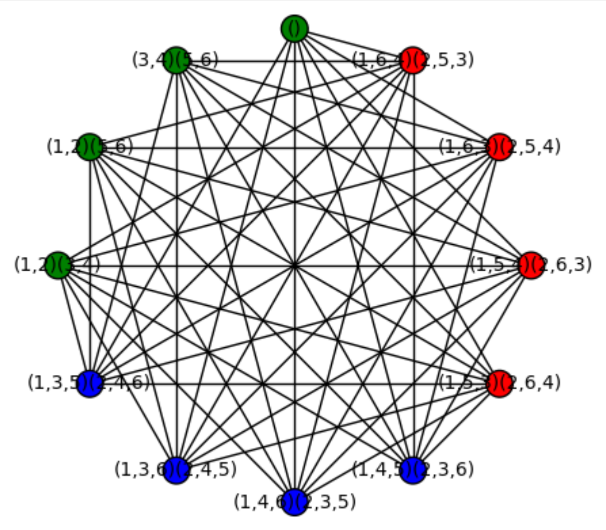

Figure: Derangement graph for $\mathrm{Alt}(4) \leqslant \mathrm{Sym}(6)$

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* Clique-coclique bound,
* Hoffman's ratio bound,

$_ The clique-coclique bound

Theorem (Clique-coclique)

Let $X = (V, E)$ be a vertex-transitive graph. Then, $\alpha(X)\omega(X) \leqslant |V(X)|$.

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- $\omega(\Gamma_G) \leqslant |\Omega|$. If equality holds, then G has the EKR property.
- If S is a sharply transitive set of G, then G has the EKR property.
- If $|\Omega|$ is a prime number, then $\rho(G) = 1$.

#_ Transitive groups with fixed degree

$_ Intersection density of groups of a given degree

For any $n \geqslant 2$, define

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Conjecture (Meagher-R-Spiga, 2021)

(I) If p is a prime number and $k \geqslant 1$, then $\mathcal{I}_{p^k} = \{1\}$.

(II) If p is an odd prime, then $\mathcal{I}_{2p} \subset [1, 2] \cap \mathbb{Q}$.

(III) If $p > q$ are two distinct odd primes, then $\mathcal{I}_{pq} = \{1\}$.

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(III) If $p > q$ are two distinct odd primes, then $\mathcal{I}_{pq} = \{1\}$.

- (I) was proved independently by Li et al., and Marušič et al., in 2021.
- (II) by R., 2021. Marušič et al. showed that $\mathcal{I}_{2p} = \{1, 2\}$.

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Theorem (Marušič et. al (2021))

There are “infinitely” many imprimitive groups of degree pq that have a unique complete block system with blocks of size q and intersection density equal to q.

Question

What about other transitive groups of degree pq?

#_ The primitive case

$_ Reduction

Lemma (No Homomorphism Lemma)

Let X and Y be two graphs, Y is vertex transitive. If $\phi : V(X) \to V(Y)$ is a homomorphism of graphs, then

$$\frac{\alpha(Y)}{|V(Y)|} \leqslant \frac{\alpha(X)}{|V(X)|}.$$

Corollary

If $H \leqslant G$ is transitive, then $\rho(G) \leqslant \rho(H)$.

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A primitive group of is either

- 2-transitive group (Meagher-Spiga-Tiep)
- simply primitive.

$_ Primitive case

- The socle $\mathrm{Soc}(G)$ of a group G is the subgroup generated by the minimal normal subgroups.
- $\mathrm{Soc}(G) \trianglelefteq G$.
- If $G \leqslant \mathrm{Sym}(\Omega)$ is primitive, then a normal subgroup is transitive or in the kernel of the action.
- If G is primitive, then $\mathrm{Soc}(G)$ is a transitive subgroup of G. In particular,

$$\rho(G) \leqslant \rho(\mathrm{Soc}(G)).$$

$_ Primitive case

Line	Socle	(p, q)	action	Intersection density
1	$\mathrm{Alt}(7)$	$(7, 5)$	triples	1
2	$\mathrm{PSL}_4(2)$	$(7, 5)$	2-spaces	1
3	$\mathrm{PSL}_5(2)$	$(31, 5)$	2-spaces	1
4	$\mathrm{PSL}_2(23)$	$(23, 11)$	cosets of $\mathrm{Sym}(4)$	1
5	$\mathrm{PSL}_2(11)$	$(11, 5)$	cosets of $\mathrm{Alt}(4)$	1
6	M_{11}	$(11, 5)$		1
7	M_{22}	$(11, 7)$		1
8	M_{23}	$(23, 11)$		1
9	$\mathrm{Alt}(p)$	$\left(p, \frac{p-1}{2}\right)$	pairs	1
10	$\mathrm{Alt}(p+1)$	$\left(p, \frac{p+1}{2}\right)$	pairs	1
11	$\mathrm{PSp}(4, k)$	$(k^2 + 1, k + 1)$	1-spaces	1
12	$P\Omega^{\varepsilon}_{2d}(2)$	$\left(2^d - \varepsilon, 2^{d-1} + \varepsilon\right)$	singular 1-spaces	???
13	$\mathrm{PSL}_2(p)$	$\left(p, \frac{p+1}{2}\right)$	cosets of D_{p-1} $p \geqslant 13$ and $p \equiv 1 (mod\ 4)$	???
14	$\mathrm{PSL}_2(p)$	$\left(p, \frac{p-1}{2}\right)$	cosets of D_{p+1} $p \geqslant 13$ and $p \equiv 3 (mod\ 4)$	1
15	$\mathrm{PSL}_2(q^2)$	$\left(\frac{q^2+1}{2}, q\right)$	cosets of $\mathrm{PGL}_2(q)$	???
16	$\mathrm{PSL}_2(p)$	$(19, 3), (29, 7), (59, 29)$	cosets of $\mathrm{Alt}(5)$	1
17	$\mathrm{PSL}_2(13)$	$(13, 7)$	cosets of $\mathrm{Alt}(4)$	1
18	$\mathrm{PSL}_2(61)$	$(61, 31)$	cosets of $\mathrm{Alt}(5)$	???

Table: Socle of simply primitive groups of degree pq.

» Weighted ratio bound.
» Clique-coclique bound.

#_ The imprimitive case

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Question

What about the imprimitive groups with a unique block system? Complete $[q^p]$-block system?

$_ Construction by Marušič et al.

» Marušič et al. constructed a $[p, k]_q$ cyclic code C, for any projective prime

$$p = \frac{q^k - 1}{q - 1} = |\,\mathrm{PG}_{k-1}(q)|,$$

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- For any $c = (c_0, c_1, \ldots, c_{p-1}) \in C$, define

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- $H(C) = \langle\{\beta_c \mid c \in C\}\rangle$ and $G(C) = \langle H(C), \alpha\rangle \cong \mathbb{F}_q^k \rtimes \mathbb{Z}_p$ is an imprimitive group of degree pq.
- $\rho(G(C)) = q$.

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Theorem (Behajaina, Maleki, R., 2022)

Let $G \leqslant \mathrm{Sym}(\Omega)$ be an imprimitive group admitting a $[q^p]$-block system and $\ker(G \to \overline{G}) \neq 1$. Then, $\rho(G) = q$ if and only if there exists a cyclic $[p, k]_q$ code C with the property that

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Theorem (Behajaina, Maleki, R., 2022)

If $p = \frac{q^k - 1}{q - 1}$ is a prime such that $k < q < p$, then there exists a transitive group of degree pq such that $\rho(G) = \frac{q}{k}$.

- We used certain permutation automorphism in $\mathrm{PAut}(C)$.

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» If $\ker(G \to \overline{G}) \neq 1$, then G is genuinely imprimitive.

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  * Any minimal normal subgroup $N \leqslant \ker(G \to \overline{G})$ is a derangement-free elementary abelian q-group.
    † If $\overline{G}$ is 2-transitive, then is $\rho(G) = 1$?
    † Is it true that if $\overline{G} < \mathrm{AGL}_1(p)$, then $\rho(G) \in \{\frac{q}{k} \ : \ k \mid (p-1) \text{ and } k < q\}$?

\$_ Open problems: primitive case

What are the intersection density of the other socles of primitive groups?

- $P\Omega_{2d}^{\varepsilon}(2)$ acting on the singular 1-spaces.
  - $\varepsilon = +$ and d is a Fermat prime.
  - $\varepsilon = -$ and $d-1$ is a Mersenne prime.
- $\mathrm{PSL}_2(p)$ acting on 2-subsets of $\mathrm{PG}_1(p)$, where $p \equiv 1 \pmod 4$.
- $\mathrm{PSL}_2(q^2)$ acting on cosets of $\mathrm{PGL}_2(q)$ (or sublines).
- $\mathrm{PSL}_2(61)$ acting on cosets of $\mathrm{Alt}(5)$.

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Thank you!!