Minimum weight of the code from intersecting lines in $\mathrm{PG}(3, q)$

Robin Simoens

Ghent University and Polytechnic University of Catalonia

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Joint work with Sam Adriaensen, Mrinmoy Datta and Leo Storme

Intro

$\mathrm{PG}(3, q)$

$\mathrm{PG}(3, q) \quad \longleftrightarrow \quad$ subspaces of $\mathbb{F}_q^4$

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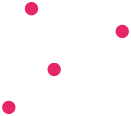
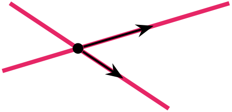
points  vector lines 

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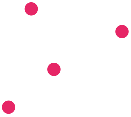
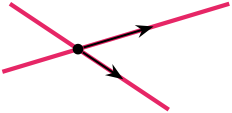
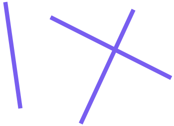
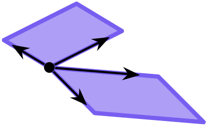
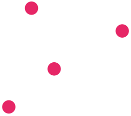
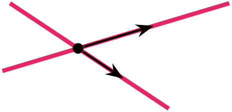
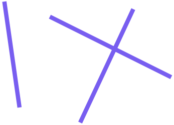
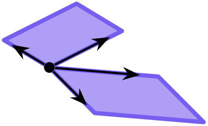
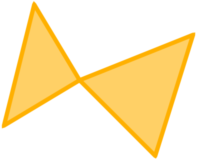
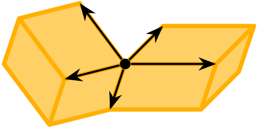
points  vector lines 

lines  vector planes 

Intro

PG$(3, q)$	$\longleftrightarrow$	subspaces of $\mathbb{F}_q^4$	
points		vector lines	
lines		vector planes	
planes		vector solids	

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Code from intersecting lines in $\mathrm{PG}(3, q)$

$$(G)_{\ell_1\ell_2} = \begin{cases} 0 & \text{if } \ell_1 \cap \ell_2 = \emptyset, \\ 1 & \text{if } \ell_1 \cap \ell_2 \neq \emptyset. \end{cases}$$

Intro

Code from intersecting lines in $\mathrm{PG}(3,q)$

$$G = \begin{array}{c c} & \begin{array}{cccc} \ell_1 & \ell_2 & \cdots & \ell_n \end{array} \\ \begin{array}{c} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix} \end{array}$$

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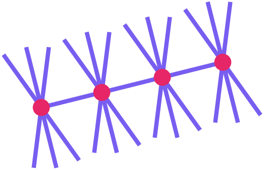
$\chi_\ell =$  has weight $q^3 + 2q^2 + q + 1$

Intro

Code from intersecting lines in PG$(3, q)$

$$G = \begin{array}{c c} & \begin{array}{cccc} \ell_1 & \ell_2 & \dots & \ell_n \end{array} \\ \begin{array}{c} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix} \end{array} \implies \mathcal{C} = \mathrm{rowspan}_{\mathbb{F}_p}(G)$$

- $n = \#$ lines in PG$(3, q) = (q^2 + 1)(q^2 + q + 1)$

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- $n = \#$ lines in $\mathrm{PG}(3, q) = (q^2+1)(q^2+q+1)$
- $k = \dim(\mathcal{C}) = \binom{p+2}{3}^h + 1$ [Goethals, Delsarte, 1968]

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Code from intersecting lines in PG$(3, q)$

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- $n = \#$ lines in PG$(3, q) = (q^2 + 1)(q^2 + q + 1)$
- $k = \dim(\mathcal{C}) = \binom{p+2}{3}^h + 1$ [Goethals, Delsarte, 1968]
- $d = ?$

Theorem

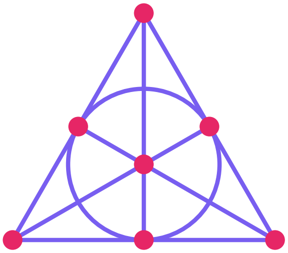
Suppose $q > 27$.

- *If q is even, then the minimum weight of $\mathcal{C}$ is $q^3 + q^2 + q + 1$.* *Minimum weight codewords are (scalar multiples of) the characteristic functions of the absolute lines of a symplectic polar space $W(3, q)$.*
- *If q is odd, then the minimum weight of $\mathcal{C}$ is strictly greater than $q^3 + q^2 + q + 1$.*

Code from points and lines

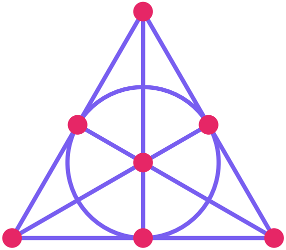
$$G = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & \dots & p_n \end{array} \\ \begin{pmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

Code from points and lines in $\mathrm{PG}(2,2)$



Incidence codes

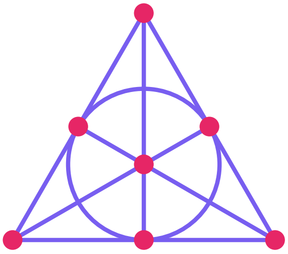
Code from points and lines in $\mathrm{PG}(2,2)$



$$
\begin{array}{c|ccccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \\
\hline
\ell_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\ell_2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\ell_3 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
\ell_4 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\ell_5 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\ell_6 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\ell_7 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

Incidence codes

Code from points and lines in PG(2, 2)

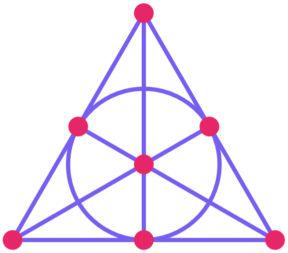


$$
\begin{array}{c|c}
 & \begin{matrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{matrix} \\
\begin{matrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \\ \ell_7 \end{matrix} &
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

$$
\begin{aligned}
C = \{&0000000, 1110000, 1001100, 1000011,\\
&0101010, 0100101, 0011001, 0010110,\\
&1101001, 1100110, 1011010, 1010101,\\
&0111100, 0110011, 0001111, 1111111\}
\end{aligned}
$$

Incidence codes

Code from points and lines in $\mathrm{PG}(2,2)$



$$
\begin{array}{c|ccccccc|}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \\
\hline
\ell_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
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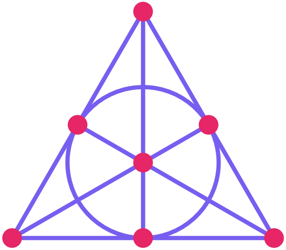
\end{aligned}
\implies [7,4,3]_2\text{-code}
$$

Incidence codes

Code from points and lines in $PG(2,2)$



$$
\begin{array}{c c}
 & \begin{array}{ccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{array} \\
\begin{array}{c} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \\ \ell_7 \end{array} &
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1 & 0 & 0 & 1 & 1 & 0 & 0 \\
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\end{pmatrix}
\end{array}
$$

$$
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Fanocode

Incidence codes

Code from points and lines in PG$(2, q)$

$$
G = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array}
\begin{array}{c} \begin{array}{cccc} p_1 & p_2 & \dots & p_n \end{array} \\
\begin{pmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \end{array}
$$

- $n = q^2 + q + 1$
- $k = \binom{p+1}{2}^h + 1$ [Graham, MacWilliams, 1966]
- $d = q + 1$ [Delsarte, Goethals, MacWilliams, 1970]

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Theorem (Szőnyi, Weiner, 2018)

Suppose $q > 27$. Let $c \neq 0$ be a codeword of the code from points and lines in $\mathrm{PG}(2, q)$. Then one of the following holds:

- $w(c) = q + 1$.
- $w(c) = 2q$.
- $w(c) = 2q + 1$.
- $w(c) \geq 3q - 3$.

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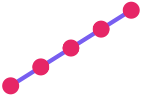

$q + 1$

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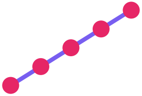
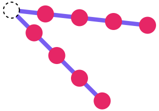


$q + 1$



$2q$

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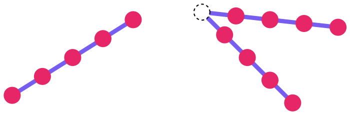

$q + 1$ $\quad$ $2q$ $\quad$ $2q + 1$

Incidence codes

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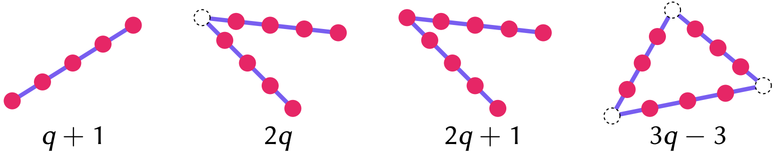

Code from lines and points

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Incidence codes

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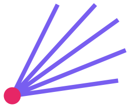

$q + 1$

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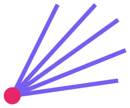
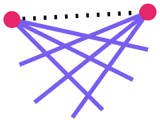


$q + 1$



$2q$

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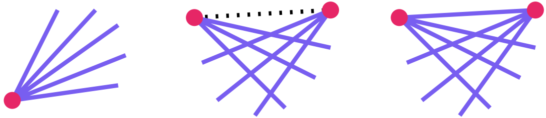

$q+1$ $2q$ $2q+1$

Incidence codes

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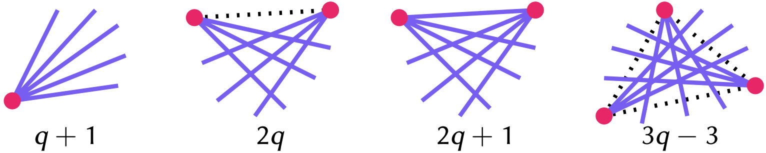


Code from intersecting lines

$$G = \begin{array}{c|c} & \begin{array}{cccc} \ell_1 & \ell_2 & \cdots & \ell_n \end{array} \\ \begin{array}{c} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix} \end{array} \implies \mathcal{C} = \mathrm{rowspan}_{\mathbb{F}_p}(G)$$

Code from intersecting lines in $\mathrm{PG}(2, q)$

$$
G = \begin{array}{c c}
 & \begin{array}{cccc} \ell_1 & \ell_2 & \cdots & \ell_n \end{array} \\
\begin{array}{c} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} & \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}
\end{array}
$$

Code from intersecting lines in $\mathrm{PG}(2, q)$

$$G = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} \begin{array}{c} \begin{array}{cccc} \ell_1 & \ell_2 & \dots & \ell_n \end{array} \\ \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \end{array} \implies \mathcal{C} = \{00\dots0, 11\dots1\}$$

Code from intersecting lines in $PG(2, q)$

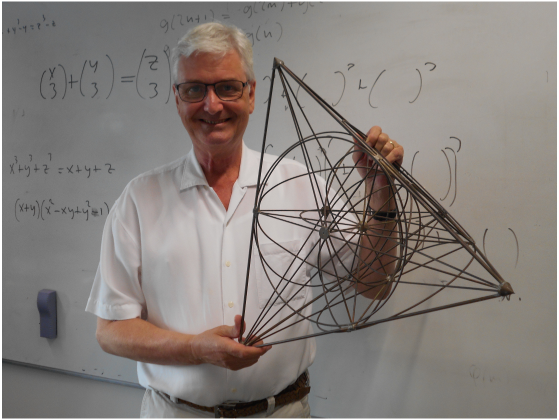
$$G = \begin{array}{c c} & \begin{array}{cccc} \ell_1 & \ell_2 & \cdots & \ell_n \end{array} \\ \begin{array}{c} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} & \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \end{array} \implies \mathcal{C} = \{00\ldots0, 11\ldots1\}$$

- $n = q^2 + q + 1$
- $k = 1$
- $d = q^2 + q + 1$

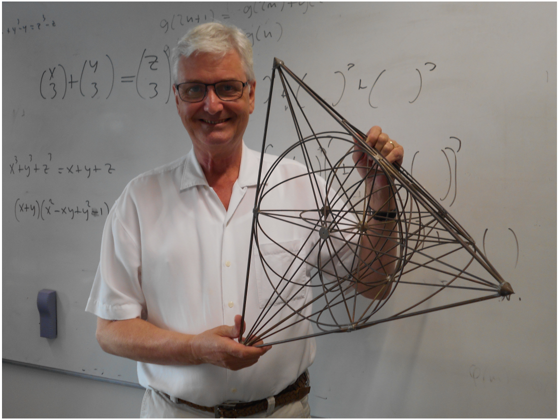
Code from intersecting lines in $\mathrm{PG}(3, q)$

$$G = \begin{array}{c c} & \begin{array}{cccc} \ell_1 & \ell_2 & \cdots & \ell_n \end{array} \\ \begin{array}{c} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix} \end{array} \implies \mathcal{C} = \mathrm{rowspan}_{\mathbb{F}_p}(G)$$

- $n = \#$ lines in $\mathrm{PG}(3, q) = (q^2 + 1)(q^2 + q + 1)$
- $k = \dim(\mathcal{C}) = \binom{p+2}{3}^h + 1$ [Goethals, Delsarte, 1968]
- $d = ?$


$\binom{x}{3}+\binom{y}{3}=\binom{z}{3}$
$x^3+y^3+z^3=x+y+z$

Code from intersecting lines in PG(3, 2)



- $[35, 7, 15]_2$-code

Lemma

If S is the set of lines in a plane π, then $c|_S$ is in the code from lines and points in π.

Code from intersecting lines in PG$(3, q)$

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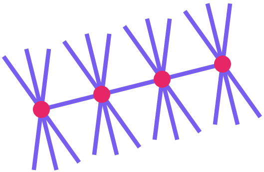
- Sufficient to prove this for "row codewords" χ_ℓ



Code from intersecting lines in $\mathrm{PG}(3, q)$

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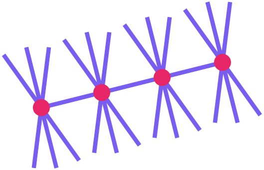

Code from intersecting lines in PG$(3, q)$

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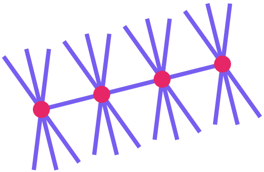

Code from intersecting lines in $\mathrm{PG}(3, q)$

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Lemma

Let $c \in \mathcal{C}$ and let S be the set of lines in a plane π. Then $c \cdot \chi_S \equiv c \cdot \mathbb{1}$.

$$c \cdot d = \sum_{\ell \in \mathcal{L}} c_\ell d_\ell$$

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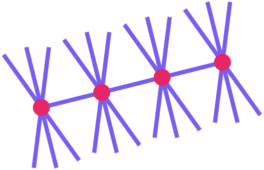

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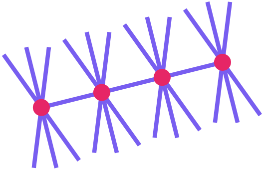

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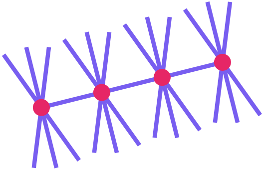

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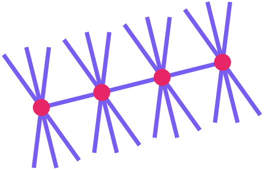

Theorem

Suppose $q > 27$. Then $w(\mathcal{C}) \geq q^3 + q^2 + q + 1$.

Code from intersecting lines in PG$(3, q)$

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Theorem

Suppose $q > 27$. Then $w(\mathcal{C}) \geq q^3 + q^2 + q + 1$.

Proof. Let $c \in \mathcal{C}$.

- $c \cdot \mathbb{1} \equiv 0 \implies w(c) > q^3 + 2q^2 + q + 1$
- $c \cdot \mathbb{1} \not\equiv 0 \implies w(c) \geq q^3 + q^2 + q + 1$

Proof (continued). $\boxed{c \cdot \mathbb{1} \equiv 0}$

Code from intersecting lines in $\mathrm{PG}(3, q)$

Proof (continued). $\boxed{c \cdot \mathbb{1} \equiv 0}$

Let S be the set of lines in a plane π

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Proof (continued). $\boxed{c \cdot \mathbb{1} \equiv 0}$

Let S be the set of lines in a plane π

- $c \cdot \chi_S \equiv 0$
- $c|_S$ is in the code from lines and points in π and $c|_S \cdot \mathbb{1} \equiv 0$

Code from intersecting lines in $\mathrm{PG}(3, q)$

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Theorem (Szőnyi, Weiner, 2018)

Suppose $q > 27$. Let $c \neq 0$ be a codeword of the code from lines and points in $\mathrm{PG}(2, q)$. Then one of the following holds:

- $w(c) = q + 1$.
- $w(c) = 2q$.
- $w(c) = 2q + 1$.
- $w(c) \geq 3q - 3$.

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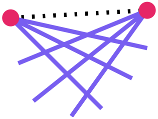
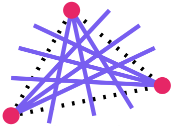
Let S be the set of lines in a plane π

- $c \cdot \chi_S \equiv 0$
- $c|_S$ is in the code from lines and points in π and $c|_S \cdot \mathbb{1} \equiv 0$
- $w(c|_S) = 0$ or $w(c|_S) = 2q$ or $w(c|_S) \geq 3q - 3$



$2q$



$3q - 3$

Proof (continued). $\boxed{c \cdot \mathbb{1} \equiv 0}$

Let S be the set of lines in a plane π

- $c \cdot \chi_S \equiv 0$
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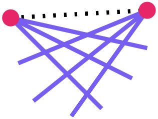

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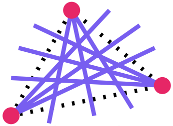

$3q - 3$

- Each plane contains 0, $2q$ or $\geq 3q - 3$ lines of $\mathrm{supp}(c)$.

Code from intersecting lines in $\mathrm{PG}(3, q)$

Proof (continued). $\boxed{c \cdot \mathbb{1} \equiv 0}$

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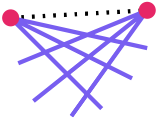
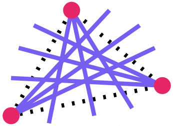
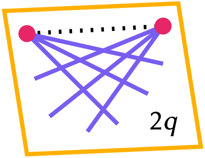




- Each plane contains 0, $2q$ or $\geq 3q - 3$ lines of $\mathrm{supp}(c)$.
- Dual: each point lies on 0, $2q$ or $\geq 3q - 3$ lines of $\mathrm{supp}(c)$.

Proof (continued). $\boxed{c \cdot \mathbb{1} \equiv 0}$

$\boxed{\text{Case 1: } \exists \text{ plane containing } 2q \text{ lines of } \mathrm{supp}(c)}$



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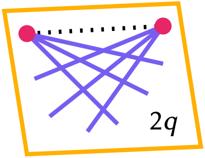
- at least $q^2(2q-2)$ other lines in $\mathrm{supp}(c)$



Code from intersecting lines in $\mathrm{PG}(3, q)$

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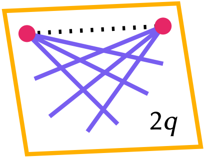

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Code from intersecting lines in PG$(3, q)$

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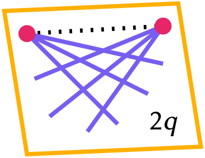

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Code from intersecting lines in $\mathrm{PG}(3, q)$

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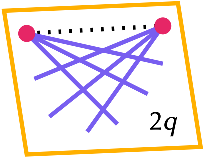

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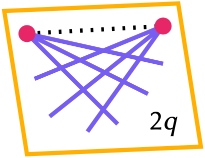
Theorem (Haemers, 1995)

A set S of lines in $\mathrm{PG}(2, q)$ covers at least $\frac{(q+1)^2|S|}{q+|S|}$ points.

Proof (continued). $\boxed{c \cdot \mathbb{1} \equiv 0}$



Case 1: $\exists$ plane containing $2q$ lines of $\mathrm{supp}(c)$

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- $\mathrm{supp}(c)$ covers at least $\frac{3}{4}q^2$ points of a plane

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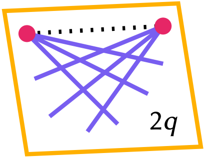

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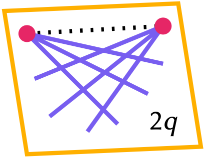
Proof (continued). $\boxed{c \cdot \mathbb{1} \equiv 0}$



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Proof (continued). $\boxed{c \cdot \mathbb{1} \not\equiv 0}$

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Code from intersecting lines in PG$(3, q)$

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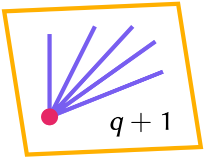

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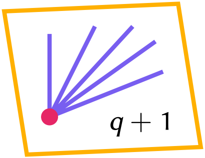

Count pairs (ℓ, π) in two ways $\implies w(c) \geq q^3 + q^2 + q + 1$

Code from intersecting lines in $\mathrm{PG}(3, q)$

Lemma

$w(c) = q^3 + q^2 + q + 1 \implies$ *c is (a scalar multiple of) the characteristic function of the absolute lines of a symplectic polar space* $W(3, q)$.

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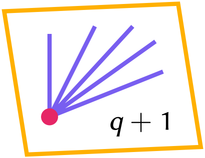
Proof idea.

- each plane π contains $q+1$ lines of $\mathrm{supp}(c)$ through a point $P(\pi)$



- $\pi \mapsto P(\pi)$ is the desired symplectic polarity

Code from intersecting lines in $\mathrm{PG}(3, q)$

Lemma

The characteristic function of the absolute lines of a symplectic space $W(3, q)$ is in the code of intersecting lines in $\mathrm{PG}(3, q) \iff q$ is even.

Code from intersecting lines in $\mathrm{PG}(3, q)$

Lemma

The characteristic function of the absolute lines of a symplectic space $W(3, q)$ is in the code of intersecting lines in $\mathrm{PG}(3, q) \iff q$ is even.

Theorem

Suppose $q > 27$.

- *If q is even, then the minimum weight of $\mathcal{C}$ is $q^3 + q^2 + q + 1$.*
  Minimum weight codewords are (scalar multiples of) the characteristic functions of the absolute lines of a symplectic polar space $W(3, q)$.
- *If q is odd, then the minimum weight of $\mathcal{C}$ is strictly greater than $q^3 + q^2 + q + 1$.*

What about odd q?

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Recall:

- $c \cdot \mathbb{1} \equiv 0 \implies w(c) > q^3 + 2q^2 + q + 1$

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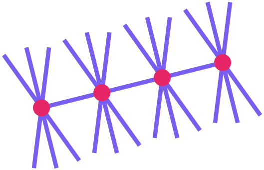
Recall:

- $c \cdot \mathbb{1} \equiv 0 \implies w(c) > q^3 + 2q^2 + q + 1$
-  has weight $q^3 + 2q^2 + q + 1$

What about odd q?

Recall:

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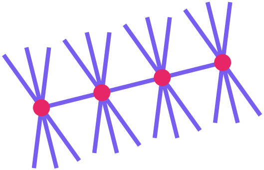
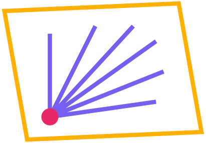
-  has weight $q^3 + 2q^2 + q + 1$

$\implies c \cdot \mathbb{1} \not\equiv 0$ is the only interesting case

What about odd q?

$\boxed{c \cdot \mathbb{1} \neq 0}$

- each plane pencil contains a line of $\mathrm{supp}(c)$



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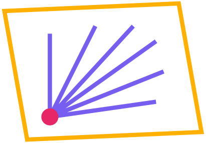

Klein correspondence

$$\mathrm{PG}(3,q) \quad \longleftrightarrow \quad Q^+(5,q)$$

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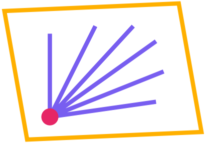

Klein correspondence

$\mathrm{PG}(3, q)$	$\longleftrightarrow$	$Q^+(5, q)$
line		point

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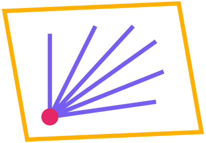

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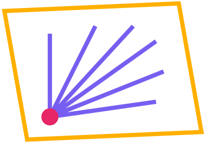

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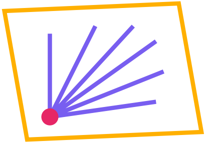

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- blocking set of $Q^+(5, q)$

What about odd q?

Theorem (Metsch, 2000)

A blocking set of $Q^+(5, q)$ with at most $q^3 + 2q^2 + q$ points contains a blocking set that is contained in a hyperplane.

What about odd q?

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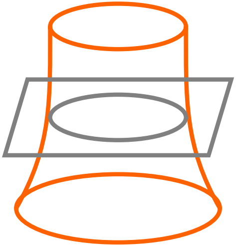

$Q(4,q)$

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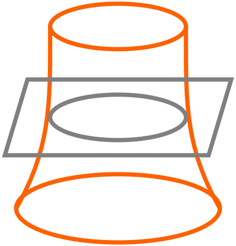
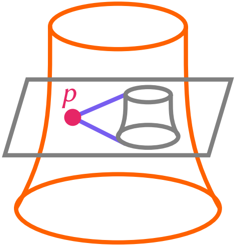


$Q(4,q)$

or



$pQ^+(3,q)$

What about odd q?

Theorem (Metsch, 2000)

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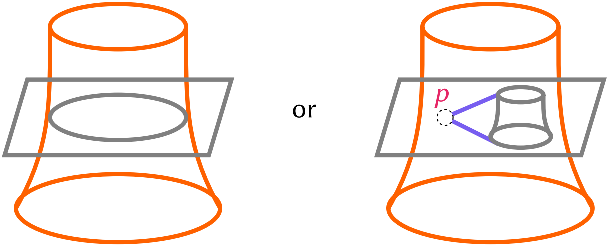

$Q(4,q)$ $\qquad$ $pQ^+(3,q)\setminus\{p\}$

What about odd q?

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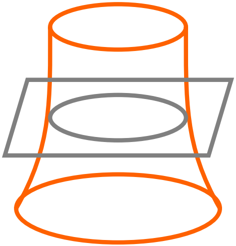

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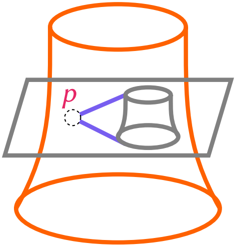

$Q(4,q)$

q^3+q^2+q+1

$pQ^+(3,q)\setminus\{p\}$

q^3+2q^2+q

What about odd q?

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Theorem

Suppose $q > 27$.

- *If q is even, then the minimum weight of $\mathcal{C}$ is $q^3 + q^2 + q + 1$. Minimum weight codewords are (scalar multiples of) the characteristic functions of the absolute lines of a symplectic polar space $W(3, q)$. The second smallest codewords have weight $q^3 + 2q^2 + q + 1$.*
- *If q is odd, then the minimum weight of $\mathcal{C}$ is strictly greater than $q^3 + q^2 + q + 1$.*

What about odd q?

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Theorem

Suppose $q > 27$.

- *If q is even, then the minimum weight of $\mathcal{C}$ is $q^3 + q^2 + q + 1$. Minimum weight codewords are (scalar multiples of) the characteristic functions of the absolute lines of a symplectic polar space $W(3, q)$. The second smallest codewords have weight $q^3 + 2q^2 + q + 1$. Are they (scalar multiples of) characteristic vectors of lines intersecting a given line?*
- *If q is odd, then the minimum weight of $\mathcal{C}$ is strictly greater than $q^3 + q^2 + q + 1$. Is it equal to $q^3 + 2q^2 + q + 1$? Are the minimum weight codewords (scalar multiples of) characteristic vectors of lines intersecting a given line?*

Thank you for listening!