

Minimum weight of the code from intersecting lines in $PG(3, q)$

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Joint work with Sam Adriaensen, Mrinmoy Datta and Leo Storme

$PG(3; q)$

$\text{PG}(3; q)$

!

subspaces of \mathbb{F}_q^4

$\text{PG}(3; q)$

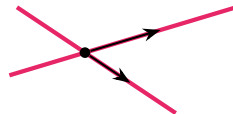
!

subspaces of \mathbb{F}_q^4

points



vector lines



$PG(3; q)$

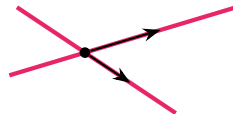
!

subspaces of F_q^4

points



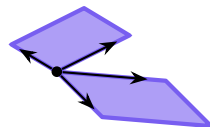
vector lines



lines



vector planes



$PG(3; q)$

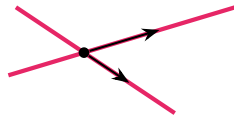
!

subspaces of F_q^4

points



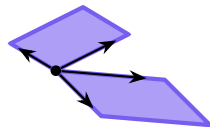
vector lines



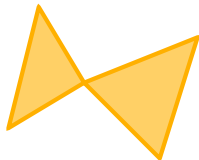
lines



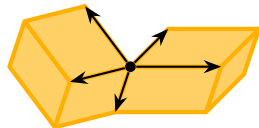
vector planes



planes



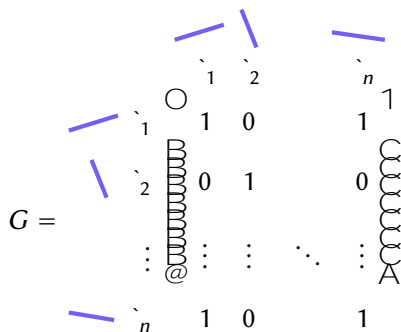
vector solids



Code from intersecting lines in $\text{PG}(3; q)$

$$(G)_{\ell_1 \ell_2} = \begin{cases} 0 & \text{if } \ell_1 \setminus \ell_2 = \emptyset \\ 1 & \text{if } \ell_1 \setminus \ell_2 \neq \emptyset \end{cases}$$

Code from intersecting lines in $PG(3; q)$



Code from intersecting lines in $\text{PG}(3; q)$

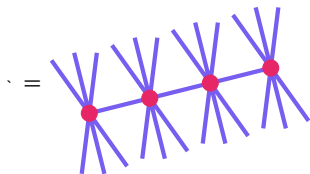
$$G = \begin{array}{c} \diagup \quad \diagdown \quad \diagup \\ \backslash \quad / \quad \backslash \\ \circ \quad \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} \quad \begin{array}{cc} 1 & 2 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \end{array} \\ \begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \end{array} \quad \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \\ \vdots \\ \diagdown \\ \diagup \\ \diagdown \\ \diagup \\ 1 \end{array} \end{array}$$

$$\Rightarrow C = \text{rowspan}_{\mathbb{F}_p}(G)$$

Code from intersecting lines in $\text{PG}(3; q)$

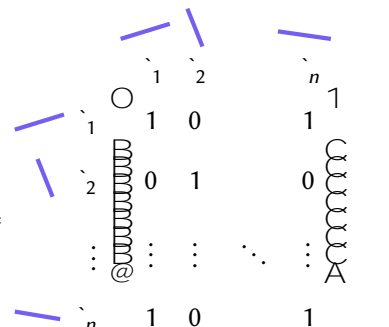
$$G = \begin{array}{c} \begin{array}{cccc} & \begin{array}{c} \diagup \\ \diagdown \end{array} & \begin{array}{c} \diagup \\ \diagdown \end{array} & \begin{array}{c} \diagup \\ \diagdown \end{array} \\ & \begin{array}{c} \diagdown \\ \diagup \end{array} & \begin{array}{c} \diagdown \\ \diagup \end{array} & \begin{array}{c} \diagdown \\ \diagup \end{array} \\ \begin{array}{c} \circ \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 1 \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 2 \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} n \\ \diagdown \\ \diagup \end{array} \\ \begin{array}{c} \diagdown \\ \diagup \end{array} & \begin{array}{c} 1 \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 0 \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 1 \\ \diagdown \\ \diagup \end{array} \\ \begin{array}{c} \diagdown \\ \diagup \end{array} & \begin{array}{c} 2 \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 0 \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 1 \\ \diagdown \\ \diagup \end{array} \\ \begin{array}{c} \vdots \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} \vdots \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} \vdots \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} \ddots \\ \diagdown \\ \diagup \end{array} \\ \begin{array}{c} \vdots \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} \vdots \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} \vdots \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} \vdots \\ \diagdown \\ \diagup \end{array} \\ \begin{array}{c} \diagdown \\ \diagup \end{array} & \begin{array}{c} n \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 1 \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 0 \\ \diagdown \\ \diagup \end{array} & \begin{array}{c} 1 \\ \diagdown \\ \diagup \end{array} \end{array} \end{array}$$

$$\Rightarrow C = \text{rowspan}_{\mathbb{F}_p}(G)$$



has weight $q^3 + 2q^2 + q + 1$

Code from intersecting lines in $\text{PG}(3; q)$


$$G = \begin{matrix} & i_1 & i_2 & \dots & i_n \\ \begin{matrix} 1 \\ 0 \\ \vdots \\ 1 \end{matrix} & 1 & 0 & \dots & 0 \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ 0 \end{matrix} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{matrix} 1 \\ 0 \\ \vdots \\ 1 \end{matrix} & 1 & 0 & \dots & 1 \end{matrix} \quad \Rightarrow \quad C = \text{rowspan}_{F_p}(G)$$

$$\ddot{a} \quad n = \# \text{ lines in } \text{PG}(3; q) = (q^2 + 1)(q^2 + q + 1)$$

Code from intersecting lines in $PG(3; q)$

$$G = \begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 \text{\textcircled{1}} & \text{\textcircled{1}} & \text{\textcircled{0}} & \text{\textcircled{1}} \\
 \text{\textcircled{2}} & \text{\textcircled{0}} & \text{\textcircled{1}} & \text{\textcircled{0}} \\
 \vdots & \vdots & \vdots & \vdots \\
 \text{\textcircled{n}} & \text{\textcircled{1}} & \text{\textcircled{0}} & \text{\textcircled{1}}
 \end{array}
 \quad \Rightarrow \quad C = \text{rowspan}_{\mathbb{F}_p}(G)$$

$$\ddot{a} \quad n = \# \text{ lines in } PG(3; q) = (q^2 + 1)(q^2 + q + 1)$$

$$\ddot{a} \quad k = \dim(C) = \binom{p+2}{3}^h + 1 \quad [\text{Goethals, Delsarte, 1968}]$$

Code from intersecting lines in $\text{PG}(3; q)$

$$G = \begin{array}{c} \\ \textcircled{\cdot} \\ \text{⌚} \\ \vdots \\ \text{⌚} \\ \text{⌚} \end{array} \begin{array}{ccc} & \begin{array}{cc} 1 & 2 \\ \vdots & \vdots \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ \vdots & \vdots \end{array} & \begin{array}{c} n \\ 1 \\ \\ \\ \end{array} \end{array}$$

The matrix G is a grid with n rows and 3 columns. The first column contains elements 1, 2, ..., n. The second column contains a grid of 1s and 0s, with a circled dot above the 1s and a clock icon to the right. The third column contains n, 1, followed by three empty rows, and then 1.

$$\Rightarrow C = \text{rowspan}_{\mathbb{F}_p}(G)$$

$$\ddot{a} \quad n = \# \text{ lines in } \text{PG}(3; q) = (q^2 + 1)(q^2 + q + 1)$$

$$\ddot{a} \quad k = \dim(C) = \binom{p+2}{3} h + 1$$

[Goethals, Delsarte, 1968]

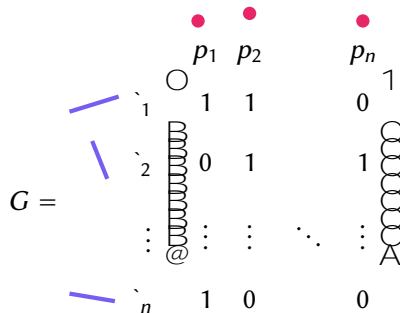
$$\ddot{a} \quad d = ?$$

Theorem

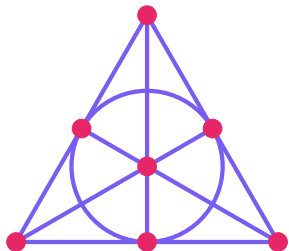
Suppose $q > 27$.

- ä If q is even, then the minimum weight of \mathcal{C} is $q^3 + q^2 + q + 1$.
Minimum weight codewords are (scalar multiples of) the characteristic functions of the absolute lines of a symplectic polar space $W(3; q)$.
- ä If q is odd, then the minimum weight of \mathcal{C} is strictly greater than $q^3 + q^2 + q + 1$.

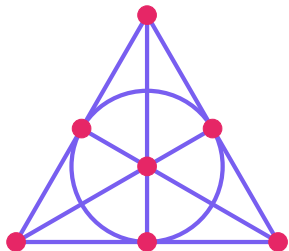
Code from points and lines



Code from points and lines in $PG(2;2)$

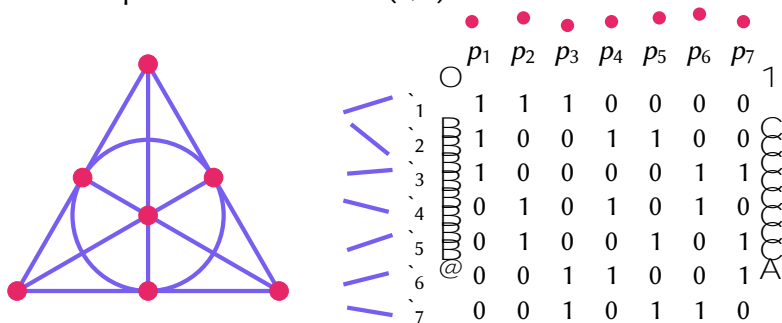


Code from points and lines in $PG(2;2)$



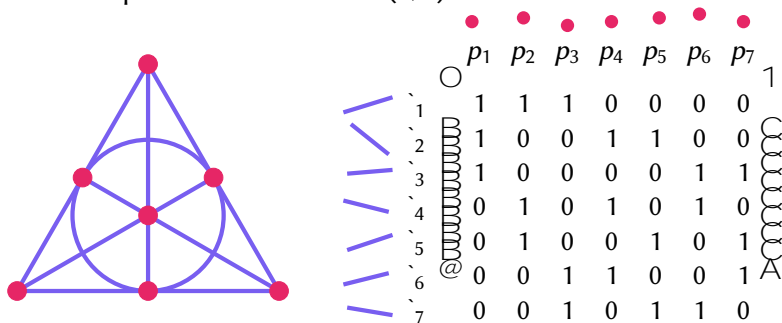
	p_1	p_2	p_3	p_4	p_5	p_6	p_7
○							1
1	1	1	1	0	0	0	0
2	1	0	0	1	1	0	0
3	1	0	0	0	0	1	1
4	0	1	0	1	0	1	0
5	0	1	0	0	1	0	1
6	0	0	1	1	0	0	1
7	0	0	1	0	1	1	0

Code from points and lines in $PG(2;2)$



$C = f0000000; 1110000; 1001100; 1000011;$
 $0101010; 0100101; 0011001; 0010110;$
 $1101001; 1100110; 1011010; 1010101;$
 $0111100; 0110011; 0001111; 1111111g$

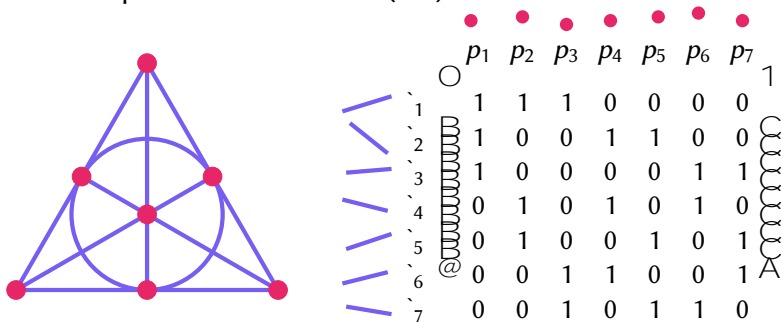
Code from points and lines in $PG(2;2)$



$C = f0000000; 1110000; 1001100; 1000011;$
 $0101010; 0100101; 0011001; 0010110;$
 $1101001; 1100110; 1011010; 1010101;$
 $0111100; 0110011; 0001111; 1111111g$

$\Rightarrow [7; 4; 3]_2$ -code

Code from points and lines in $PG(2;2)$

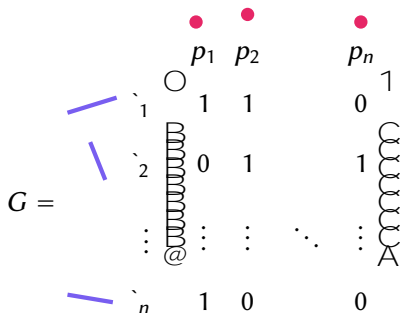


$C = f0000000; 1110000; 1001100; 1000011;$
 $0101010; 0100101; 0011001; 0010110;$
 $1101001; 1100110; 1011010; 1010101;$
 $0111100; 0110011; 0001111; 1111111g$

$\Rightarrow [7; 4; 3]_2$ -code

Fanocode

Code from points and lines in $PG(2; q)$



$$\ddot{a} \quad n = q^2 + q + 1$$

$$\ddot{a} \quad k = \binom{q+1}{2} + 1$$

$$\ddot{a} \quad d = q + 1$$

[Graham, MacWilliams, 1966]

[Delsarte, Goethals, MacWilliams, 1970]

Theorem (Szőnyi, Weiner, 2018)

Suppose $q > 27$. Let $c \neq 0$ be a codeword of the code from points and lines in $\text{PG}(2; q)$. Then one of the following holds:

• $w(c) = q + 1$.

• $w(c) = 2q$.

• $w(c) = 2q + 1$.

• $w(c) = 3q + 3$.

Theorem (Szőnyi, Weiner, 2018)

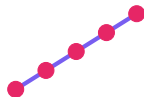
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$$q + 1$$

Theorem (Szőnyi, Weiner, 2018)

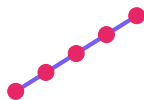
Suppose $q > 27$. Let $c \notin 0$ be a codeword of the code from points and lines in $\text{PG}(2; q)$. Then one of the following holds:

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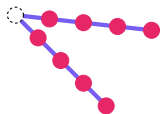
• $w(c) = 2q$.

• $w(c) = 2q + 1$.

• $w(c) = 3q - 3$.



$$q + 1$$



$$2q$$

Theorem (Szőnyi, Weiner, 2018)

Suppose $q > 27$. Let $c \notin 0$ be a codeword of the code from points and lines in $\text{PG}(2; q)$. Then one of the following holds:

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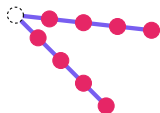
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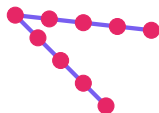
• $w(c) = 3q + 3$.



$q + 1$



$2q$



$2q + 1$

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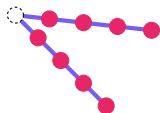
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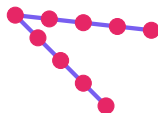
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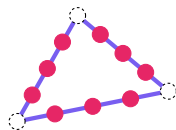
$q + 1$



$2q$

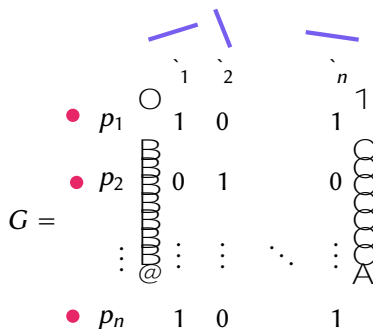


$2q + 1$



$3q + 3$

Code from lines and points



Theorem (Szőnyi, Weiner, 2018)

Suppose $q > 27$. Let $c \neq 0$ be a codeword of the code from lines and points in $\text{PG}(2; q)$. Then one of the following holds:

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Theorem (Szőnyi, Weiner, 2018)

Suppose $q > 27$. Let $c \neq 0$ be a codeword of the code from lines and points in $\text{PG}(2; q)$. Then one of the following holds:

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$$q + 1$$

Theorem (Szőnyi, Weiner, 2018)

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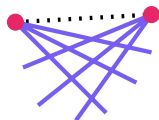
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$$q + 1$$



$$2q$$

Theorem (Szőnyi, Weiner, 2018)

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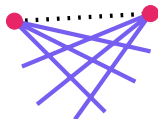
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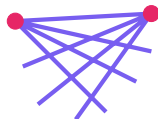
• $w(c) = 3q + 3$.



$$q + 1$$



$$2q$$



$$2q + 1$$

Theorem (Szőnyi, Weiner, 2018)

Suppose $q > 27$. Let $c \notin 0$ be a codeword of the code from lines and points in $\text{PG}(2; q)$. Then one of the following holds:

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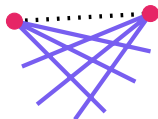
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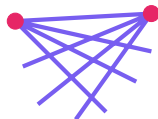
• $w(c) = 3q + 3$.



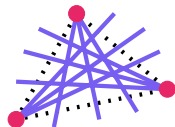
$q + 1$



$2q$



$2q + 1$



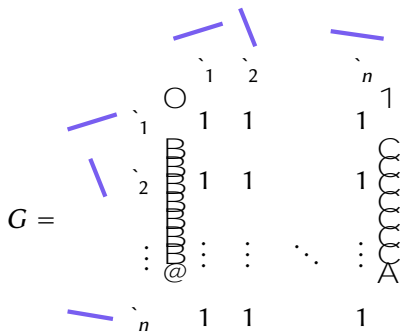
$3q + 3$

Code from intersecting lines

$$G = \begin{array}{c|ccc} & \begin{array}{c} \text{1} \\ \text{2} \\ \vdots \\ \text{n} \end{array} & \begin{array}{c} \text{1} \\ \text{2} \\ \vdots \\ \text{n} \end{array} & \begin{array}{c} \text{1} \\ \text{2} \\ \vdots \\ \text{n} \end{array} \\ \hline \text{1} & 1 & 0 & 1 \\ \text{2} & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \text{n} & 1 & 0 & 1 \end{array}$$

$$\Rightarrow C = \text{rowspan}_{\mathbb{F}_p}(G)$$

Code from intersecting lines in $PG(2, q)$



Code from intersecting lines in $PG(2, q)$

$$G = \begin{array}{cccc}
 & & \text{---} & \text{---} \\
 & & \backslash & \backslash \\
 & & \text{' } 1 & \text{' } 2 \\
 & & \text{O} & \\
 \text{' } 1 & & 1 & 1 \\
 \text{' } 2 & & 1 & 1 \\
 \vdots & & \vdots & \vdots \\
 \text{' } n & & 1 & 1 \\
 & & \text{---} & \text{---} \\
 & & \text{A} & \\
 & & \text{---} & \text{---}
 \end{array}$$

$$\Rightarrow C = f00 \dots 0; 11 \dots 1g$$

Code from intersecting lines in $PG(2, q)$

$$G = \begin{array}{c} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \\ \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \text{---} & \text{---} \end{array} \\ \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \text{---} & \text{---} \end{array} \\ \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \text{---} & \text{---} \end{array} \end{array}$$

$$\Rightarrow C = f00 \dots 0; 11 \dots 1g$$

ä $n = q^2 + q + 1$

ä $k = 1$

ä $d = q^2 + q + 1$

Code from intersecting lines in $\text{PG}(3, q)$

$$G = \begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 \circ & 1 & 2 & \\
 \vdots & \vdots & \vdots & \ddots \\
 \text{A} & 1 & 0 & 1
 \end{array}
 \Rightarrow C = \text{rowspan}_{\mathbb{F}_p}(G)$$

ä $n = \# \text{ lines in } \text{PG}(3; q) = (q^2 + 1)(q^2 + q + 1)$

ä $k = \dim(C) = \binom{p+2}{3}^h + 1$ [Goethals, Delsarte, 1968]

ä $d = ?$

Code from intersecting lines in $PG(3, 2)$

Code from intersecting lines in $PG(3, 2)$

• $[35; 7; 15]_2$ -code

Lemma

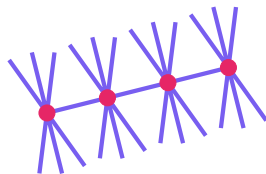
If S is the set of lines in a plane π , then c_{j_S} is in the code from lines and points in π .

Lemma

If S is the set of lines in a plane π , then c_{j_S} is in the code from lines and points in π .

Proof.

ä Sufficient to prove this for “row codewords”



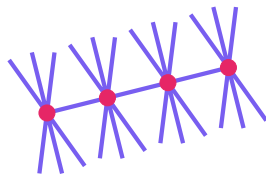
Lemma

If S is the set of lines in a plane π , then c_j^S is in the code from lines and points in π .

Proof.

It is sufficient to prove this for “row codewords”.

Let π be a plane in $PG(3, q)$. Then π contains $q^2 + q + 1$ lines. Let S be the set of lines in π . Then c_j^S is in the code from lines and points in π .



Lemma

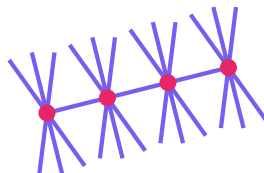
If S is the set of lines in a plane, then c_{jS} is in the code from lines and points in \mathbb{P}^3 .

Proof.

It is sufficient to prove this for “row codewords”.

$$c_j = \sum_{i=1}^p \lambda_i \cdot \text{row}_i \Rightarrow \sum_{i=1}^p \lambda_i = 1$$

$$c_j = \sum_{i=1}^p \lambda_i \cdot \text{row}_i \Rightarrow \sum_{i=1}^p \lambda_i = 1$$



Code from intersecting lines in $\mathbb{P}G(3; q)$

Lemma

Let $c \in C$ and let S be the set of lines in a plane. Then $c \cdot S = c \cdot 1$.

$$c \cdot d = \sum_{l \in S} c \cdot d$$

Code from intersecting lines in $\mathbb{P}G(3; q)$

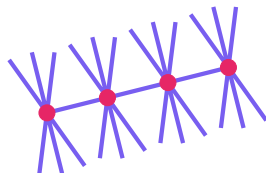
Lemma

Let $c \in C$ and let S be the set of lines in a plane. Then $c \cdot s = c + 1$.

$$c \cdot d = \sum_{L \in \mathcal{L}} c \cdot d$$

Proof.

It is sufficient to prove this for "row codewords".



Code from intersecting lines in $\mathbb{P}G(3; q)$

Lemma

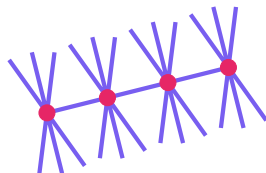
Let $c \in \mathbb{C}$ and let S be the set of lines in a plane. Then $c \cdot |S| = c + 1$.

$$c \cdot d = \sum_{L \in S} c \cdot d_L$$

Proof.

It is sufficient to prove this for "row codewords".

$$c \cdot (q^3 + 2q + q + 1) = c + 1$$



Code from intersecting lines in $\mathbb{P}G(3; q)$

Lemma

Let $c \in C$ and let S be the set of lines in a plane. Then $c \cdot s = c + 1$.

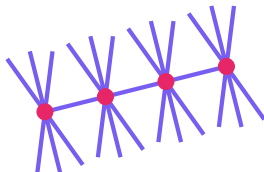
$$c \cdot d = \sum_{d \in \mathbb{P}^2} c \cdot d$$

Proof.

It is sufficient to prove this for "row codewords".

$$c \cdot 1 = q^3 + 2q + q + 1 = 1$$

$$\Rightarrow c \cdot s = q^2 + q + 1 = 1$$



Code from intersecting lines in $\mathbb{P}G(3; q)$

Lemma

Let $c \in \mathbb{C}$ and let S be the set of lines in a plane. Then $c \cdot s = c + 1$.

$$c \cdot d = \sum_{d \in \mathbb{P}^2} c \cdot d$$

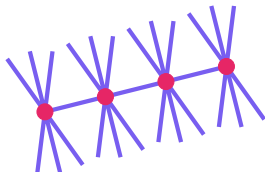
Proof.

It is sufficient to prove this for "row codewords".

$$c \cdot 1 = q^3 + 2q + q + 1 = 1$$

$$c \cdot s = q^2 + q + 1 = 1$$

$$c \cdot \setminus = \text{fpg} \Rightarrow c \cdot s = q + 1 = 1$$



Code from intersecting lines in $\mathbb{P}G(3; q)$

Theorem

Suppose $q \neq 27$. Then $w(C) = q^3 + q^2 + q + 1$.

Code from intersecting lines in $\mathbb{P}G(3; q)$

Theorem

Suppose $q \geq 27$. Then $w(C) = q^3 + q^2 + q + 1$.

Proof. Let $c \in C$.

- $c \perp 1 = 0 \Rightarrow w(c) > q^3 + 2q^2 + q + 1$
- $c \perp 16 = 0 \Rightarrow w(c) = q^3 + q^2 + q + 1$

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued)

c	1	0
---	---	---

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = 1 \quad 0$

Let S be the set of lines in a plane

$\#S = c = 0$

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = 1 \quad 0$

Let S be the set of lines in a plane

$$\bar{a} = c_s = 0$$

$\bar{a} = c_s$ is in the code from lines and points in \bar{a} and $c_s = 1 \quad 0$

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = 1 \quad 0$

Let S be the set of lines in a plane

$$\# S = q + 1$$

c_j is in the code from lines and points in $\mathbb{P}G(3; q)$ and $c_j = 1 \quad 0$

Theorem (Sanyal, Weiner, 2018)

Suppose $q \geq 27$. Let $c \in \mathbb{F}_q^6$ be a codeword of the code from lines and points in $\mathbb{P}G(2; q)$. Then one of the following holds:

$$\text{a) } w(c) = q + 1.$$

$$\text{b) } w(c) = 2q.$$

$$\text{c) } w(c) = 2q + 1.$$

$$\text{d) } w(c) = 3q - 3.$$

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = (1 \ 0)$

Let S be the set of lines in a plane

$$\# S = q^2 + q + 1$$

$\# S$ is in the code from lines and points in $\mathbb{P}G(3; q)$ and $\# S = q^2 + q + 1$

Theorem (Szyni, Weiner, 2018)

Suppose $q \geq 27$. Let $c \in \mathbb{F}_q^{\# S}$ be a codeword of the code from lines and points in $\mathbb{P}G(2; q)$. Then one of the following holds:

$$\text{a) } w(c) = q + 1. \Rightarrow c = (1 \ 0 \ 0)$$

$$\text{b) } w(c) = 2q.$$

$$\text{c) } w(c) = 2q + 1. \Rightarrow c = (1 \ 0 \ 0)$$

$$\text{d) } w(c) = 3q - 3.$$

Code from intersecting lines in $\mathbb{P}G(3; q)$

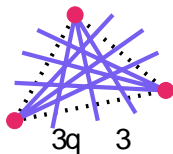
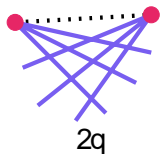
Proof (continued) $c = 1 \quad 0$

Let S be the set of lines in a plane

$\bar{a} \quad c = s = 0$

$\bar{a} \quad c_j S$ is in the code from lines and points in and $c_j S = 1 \quad 0$

$\bar{a} \quad w(c_j S) = 0$ or $w(c_j S) = 2q$ or $w(c_j S) = 3q - 3$



Code from intersecting lines in $\mathbb{P}G(3; q)$

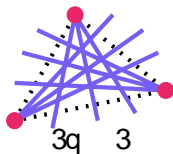
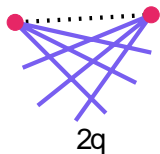
Proof (continued) $c = 1 \quad 0$

Let S be the set of lines in a plane

$$\# S = 2q + 1$$

c_S is in the code from lines and points in $\mathbb{P}G(3; q)$ and $c_S = 1 \quad 0$

$$w(c_S) = 0 \text{ or } w(c_S) = 2q \text{ or } w(c_S) = 3q - 3$$



Each plane contains $0, 2q$ or $3q - 3$ lines of $\text{supp}(c)$.

Code from intersecting lines in $\mathbb{P}G(3; q)$

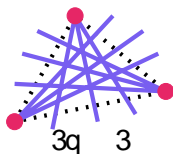
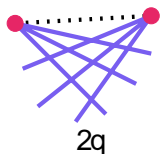
Proof (continued) $c = 1 \quad 0$

Let S be the set of lines in a plane

$$\# S = 2q + 1$$

c_S is in the code from lines and points in $\mathbb{P}G(3; q)$ and $c_S = 1 \quad 0$

$$w(c_S) = 0 \text{ or } w(c_S) = 2q \text{ or } w(c_S) = 3q - 3$$



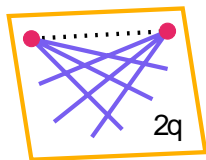
Each plane contains $0, q$ or $3q - 3$ lines of $\text{supp}(c)$.

Dual: each point lies on $0, q$ or $3q - 3$ lines of $\text{supp}(c)$.

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c \begin{matrix} 1 & 0 \end{matrix}$

Case 19 plane containing $2q$ lines of $\text{supp}(c)$

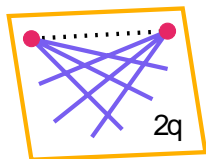


Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = 1 \quad 0$

Case 1: plane containing $2q$ lines of $\text{supp}(c)$

• at least $q^2(2q - 2)$ other lines in $\text{supp}(c)$

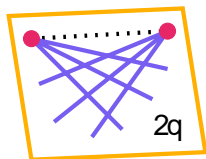


Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = 1 \quad 0$

Case 1: plane containing $2q$ lines of $\text{supp}(c)$

- at least $q^2(2q - 2)$ other lines in $\text{supp}(c)$
- $w(c) = q^2(2q - 2) > q^3 + 2q^2 + q + 1$



Code from intersecting lines in $\mathbb{P}G(3; q)$

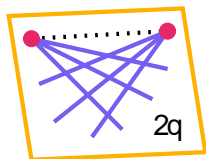
Proof (continued) $c = 1 \quad 0$

Case 19 plane containing $2q$ lines of $\text{supp}(c)$

• at least $q^2(2q - 2)$ other lines in $\text{supp}(c)$

• $w(c) = q^2(2q - 2) > q^3 + 2q^2 + q + 1$

Case 28 plane contains 0 or $3q - 3$ lines of $\text{supp}(c)$



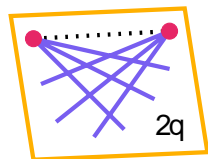
Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = 1 \quad 0$

Case 19 plane containing $2q$ lines of $\text{supp}(c)$

• at least $q^2(2q - 2)$ other lines in $\text{supp}(c)$

• $w(c) = q^2(2q - 2) > q^3 + 2q^2 + q + 1$



Case 28 plane contains 0 or $3q - 3$ lines of $\text{supp}(c)$

Theorem (Haemers, 1995)

A set S of lines in $\mathbb{P}G(2; q)$ covers at least $\frac{(q+1)^2 |S|}{q+|S|}$ points.

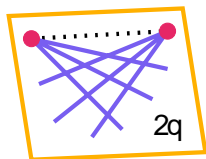
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Proof (continued) $c = 1 \quad 0$

Case 19 plane containing $2q$ lines of $\text{supp}(c)$

• at least $q^2(2q - 2)$ other lines in $\text{supp}(c)$

• $w(c) = q^2(2q - 2) > q^3 + 2q^2 + q + 1$



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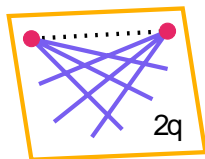
• $\text{supp}(c)$ covers at least $\frac{3}{4}q^2$ points of a plane

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = 1 \quad 0$

Case 19 plane containing $2q$ lines of $\text{supp}(c)$

- at least $q^2(2q - 2)$ other lines in $\text{supp}(c)$
- $w(c) = q^2(2q - 2) > q^3 + 2q^2 + q + 1$



Case 28 plane contains 0 or $3q - 3$ lines of $\text{supp}(c)$

Theorem (Haemers, 1995)

A set S of lines in $\mathbb{P}G(2; q)$ covers at least $\frac{(q+1)^2 |S|}{q+|S|}$ points.

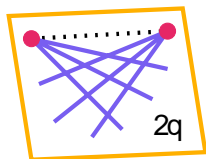
- $\text{supp}(c)$ covers at least $\frac{3}{4}q^2$ points of a plane
- at least $\frac{3}{4}q^2(3q - 3 - (q + 1))$ lines in $\text{supp}(c)$

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) $c = 1, 0$

Case 19 plane containing $2q$ lines of $\text{supp}(c)$

- at least $q^2(2q - 2)$ other lines in $\text{supp}(c)$
- $w(c) = q^2(2q - 2) > q^3 + 2q^2 + q + 1$



Case 28 plane contains 0 or $3q - 3$ lines of $\text{supp}(c)$

Theorem (Haemers, 1995)

A set S of lines in $\mathbb{P}G(2; q)$ covers at least $\frac{(q+1)^2 |S|}{q+|S|}$ points.

- $\text{supp}(c)$ covers at least $\frac{3}{4}q^2$ points of a plane
- at least $\frac{3}{4}q^2(3q - 3 - (q + 1))$ lines in $\text{supp}(c)$
- $w(c) > q^3 + 2q^2 + q + 1$

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) c 1 6 0

Code from intersecting lines in $\mathbb{P}G(3; q)$

Proof (continued) c 1 6 0

Let S be the set of lines in a plane

$\tilde{a} \quad c \quad s \quad 6 \quad 0$

Proof (continued). $\boxed{c \ 1 \ 6 \ 0}$

Let S be the set of lines in a plane

$\bar{a} \ c \ s \ 6 \ 0$

$\bar{a} \ c/s$ is in the code from lines and points in S and $c/s \ 1 \ 6 \ 0$

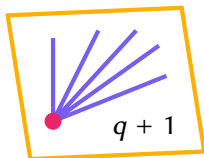
Proof (continued). $\boxed{c \ 1 \ 6 \ 0}$

Let S be the set of lines in a plane

\ddot{a} $c \quad s \ 6 \ 0$

\ddot{a} c/s is in the code from lines and points in \quad and $c/s \ 1 \ 6 \ 0$

\ddot{a} $w(c/s) \quad q + 1$



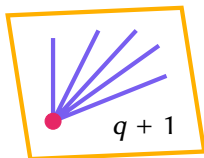
Proof (continued). $\boxed{c \ 1 \ 6 \ 0}$

Let S be the set of lines in a plane

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\bar{a} c/s is in the code from lines and points in \quad and $c/s \ 1 \ 6 \ 0$

\bar{a} $w(c/s) \quad q + 1$



Count pairs $(\cdot; \cdot)$ in two ways $\Rightarrow w(c) \quad q^3 + q^2 + q + 1$

Lemma

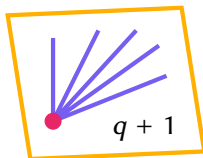
$w(c) = q^3 + q^2 + q + 1 \Rightarrow c$ is (a scalar multiple of) the characteristic function of the absolute lines of a symplectic polar space $W(3; q)$.

Lemma

$w(c) = q^3 + q^2 + q + 1 \Rightarrow c$ is (a scalar multiple of) the characteristic function of the absolute lines of a symplectic polar space $W(3; q)$.

Proof idea.

• each plane contains $q + 1$ lines of $\text{supp}(c)$ through a point $P(\)$



• $\nabla P(\)$ is the desired symplectic polarity

Lemma

The characteristic function of the absolute lines of a symplectic space $W(3; q)$ is in the code of intersecting lines in $\text{PG}(3; q)$ () q is even.

Lemma

The characteristic function of the absolute lines of a symplectic space $W(3; q)$ is in the code of intersecting lines in $\text{PG}(3; q)$ (\quad) q is even.

Theorem

Suppose $q > 27$.

- ä If q is even, then the minimum weight of \mathcal{C} is $q^3 + q^2 + q + 1$.
Minimum weight codewords are (scalar multiples of) the characteristic functions of the absolute lines of a symplectic polar space $W(3; q)$.*
- ä If q is odd, then the minimum weight of \mathcal{C} is strictly greater than $q^3 + q^2 + q + 1$.*

What about odd q ?

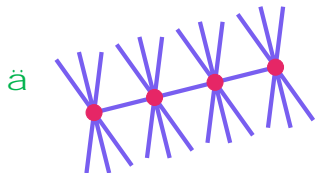
Recall:

$$\tilde{a}(c-1) = 0 \Rightarrow w(c) > q^3 + 2q^2 + q + 1$$

What about odd q ?

Recall:

$$\ddot{a} \quad c \quad 1 \quad 0 \Rightarrow w(c) > q^3 + 2q^2 + q + 1$$

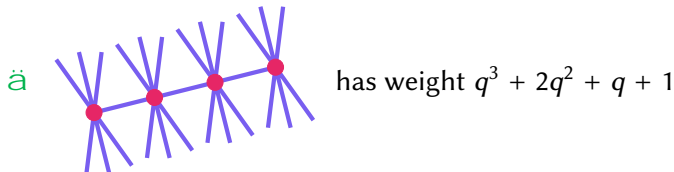


has weight $q^3 + 2q^2 + q + 1$

What about odd q ?

Recall:

$$\ddot{a} \quad c \quad 1 \quad 0 \Rightarrow w(c) > q^3 + 2q^2 + q + 1$$

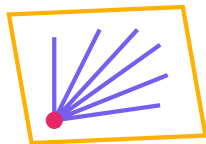


$\Rightarrow c \quad 1 \quad 0$ is the only interesting case

What about odd q ?

$$c \ 1 \ 6 \ 0$$

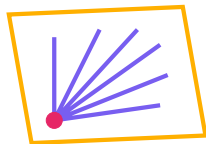
ä each plane pencil contains a line of $\text{supp}(c)$



What about odd q ?

$$c \ 1 \ 6 \ 0$$

ä each plane pencil contains a line of $\text{supp}(c)$



Klein correspondence

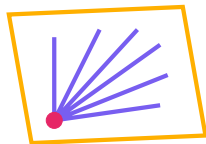
$$\text{PG}(3; q) \quad ! \quad Q^+(5; q)$$



What about odd q ?

$c \ 1 \ 6 \ 0$

ä each plane pencil contains a line of $\text{supp}(c)$



Klein correspondence

$\text{PG}(3; q) \quad ! \quad Q^+(5; q)$

line

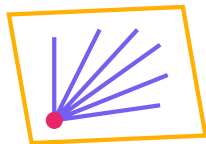
point



What about odd q ?

$$c \ 1 \ 6 \ 0$$

ä each plane pencil contains a line of $\text{supp}(c)$



Klein correspondence

$$\text{PG}(3; q) \quad ! \quad Q^+(5; q)$$



line

point

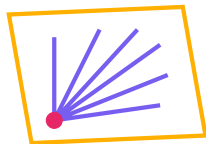
intersecting lines

collinear points

What about odd q ?

$c \ 1 \ 6 \ 0$

ä each plane pencil contains a line of $\text{supp}(c)$



Klein correspondence

$\text{PG}(3; q) \quad ! \quad Q^+(5; q)$



line

point

intersecting lines

collinear points

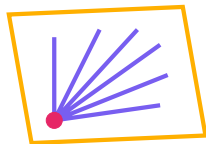
plane pencil

line

What about odd q ?

$c \ 1 \ 6 \ 0$

ä each plane pencil contains a line of $\text{supp}(c)$



Klein correspondence

$\text{PG}(3; q) \quad ! \quad Q^+(5; q)$



line	point
intersecting lines	collinear points
plane pencil	line

ä blocking set of $Q^+(5; q)$

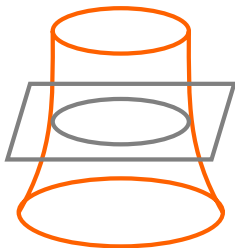
What about odd q ?

Theorem (Metsch, 2000)

A blocking set of $Q^+(5; q)$ with at most $q^3 + 2q^2 + q$ points contains a blocking set that is contained in a hyperplane.

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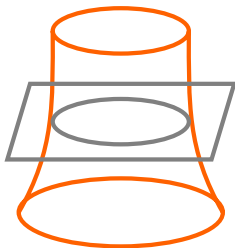


$Q(4; q)$

What about odd q ?

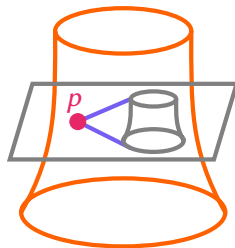
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$Q(4; q)$

or

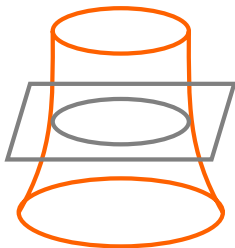


$pQ^+(3; q)$

What about odd q ?

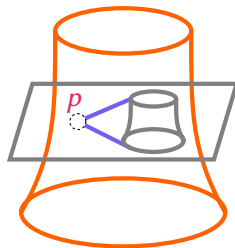
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$Q(4; q)$

or

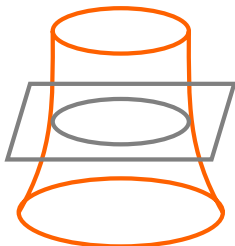


$pQ^+(3; q) \cap \text{plane}$

What about odd q ?

Theorem (Metsch, 2000)

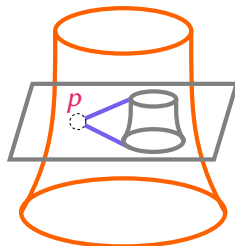
A blocking set of $Q^+(5; q)$ with at most $q^3 + 2q^2 + q$ points contains a blocking set that is contained in a hyperplane.



$Q(4; q)$

$$q^3 + q^2 + q + 1$$

or



$pQ^+(3; q) \cap \text{fp}g$

$$q^3 + 2q^2 + q$$

Theorem

Suppose $q > 27$.

- ä If q is even, then the minimum weight of \mathcal{C} is $q^3 + q^2 + q + 1$. Minimum weight codewords are (scalar multiples of) the characteristic functions of the absolute lines of a symplectic polar space $W(3; q)$. The second smallest codewords have weight $q^3 + 2q^2 + q + 1$.
- ä If q is odd, then the minimum weight of \mathcal{C} is strictly greater than $q^3 + q^2 + q + 1$.

Theorem

Suppose $q > 27$.

- ä If q is even, then the minimum weight of \mathcal{C} is $q^3 + q^2 + q + 1$. Minimum weight codewords are (scalar multiples of) the characteristic functions of the absolute lines of a symplectic polar space $W(3; q)$. The second smallest codewords have weight $q^3 + 2q^2 + q + 1$. Are they (scalar multiples of) characteristic vectors of lines intersecting a given line?
- ä If q is odd, then the minimum weight of \mathcal{C} is strictly greater than $q^3 + q^2 + q + 1$. Is it equal to $q^3 + 2q^2 + q + 1$? Are the minimum weight codewords (scalar multiples of) characteristic vectors of lines intersecting a given line?

Thank you for listening!