On regular systems of finite classical polar spaces

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m-regular systems

Finite classical polar spaces

Let $\mathcal{P}$ be a finite classical polar space. Hence $\mathcal{P}$ is a member of one of the following classes: a symplectic space $W(2n+1, q)$, a parabolic quadric $Q(2n, q)$, an hyperbolic quadric $Q^+(2n+1, q)$, an elliptic quadric $Q^-(2n+1, q)$ or an Hermitian variety $H(n, q)$ (q a square). A projective subspace of maximal dimension contained in $\mathcal{P}$ is called a *generator* of $\mathcal{P}$. The vector dimension of a generator of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$. $\mathcal{P}_{d,e}$ will denote a polar space of rank $d \geq 2$ as follows:

$\mathcal{P}_{d,e}$	$Q^+(2d-1, q)$	$H(2d-1, q)$	$W(2d-1, q)$	$Q(2d, q)$	$H(2d, q)$	$Q^-(2d+1, q)$
e	0	1/2	1	1	3/2	2

$\mathcal{M}_{\mathcal{P}_{d,e}}$ will denote the set of generators of the polar space $\mathcal{P}_{d,e}$, while $|\mathcal{M}_{\mathcal{P}_{d-k,e}}|$ will denote the number of generators passing through a $(k-1)$-space.

m-regular systems

Historical background

Definition

An m-regular system on a polar space $\mathcal{P}_{d,e}$ is a set $\mathcal{R}$ of generators such that every point of $\mathcal{P}_{d,e}$ lies on exactly m generators in $\mathcal{R}$, $0 \leq m \leq |\mathcal{M}_{\mathcal{P}_{d-1,e}}|$.

m-regular systems were introduced on Hermitian varieties in 1965 by Beniamino Segre in *Forme e geometrie hermitiane, con particolare riguardo al caso finito*. In that article Segre proved the following theorem on Hermitian surfaces $H(3, q^2)$, whose generators are lines, and each point lies on $n = q + 1$ of them.

Theorem (Segre's Theorem)

Let $\mathcal{H} = H(3, q^2)$ be an Hermitian surface. If q is odd, all the m-regular systems on $\mathcal{H}$ are hemistystems, i.e. $m = \frac{n}{2} = \frac{q+1}{2}$.

m-regular systems

Known facts on regular systems

Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be an m-regular system and an m'-regular system of $\mathcal{P}_{d,e}$, respectively, then:

1. $|\mathcal{A}| = m(q^{d+e-1}+1)$, $|\mathcal{B}| = m'(q^{d+e-1}+1)$;
2. $\mathcal{M}_{\mathcal{P}_{d,e}} \setminus \mathcal{A}$ *is also a* $\widetilde{m}$*-regular system,* $\widetilde{m} = |\mathcal{M}_{\mathcal{P}_{d-1,e}}| - m$ *(and analogously for* $\mathcal{B}$*);*
3. *if* $\mathcal{A} \subseteq \mathcal{B}$*, then* $\mathcal{B} \setminus \mathcal{A}$ *is an* $(m'-m)$*-regular system;*
4. *if* $\mathcal{A}$ *and* $\mathcal{B}$ *are disjoint, then* $\mathcal{A} \cup \mathcal{B}$ *is an* $(m+m')$*-regular system;*
5. *the empty set and* $\mathcal{M}_{\mathcal{P}_{d,e}}$ *are trivial examples of regular systems,* $m = 0, |\mathcal{M}_{\mathcal{P}_{d-1,e}}|$*, respectively.*

m-regular systems arising from field reduction

Field reduction map

Let $K = GF(q)$ and $L = GF(q^2)$. All elements of L are seen as 2-dimensional vectors over K. Taking $\omega \in L \setminus K$, the set $\{1, \omega\}$ is a basis of L over K, and for $\lambda \in L$:

$$\lambda = \lambda_1 + \omega\lambda_2$$

with λ_1, $\lambda_2 \in K$.
The image under the *field reduction* map ϕ of an r-dimensional vector space over $GF(q^2)$ is a $2r$-dimensional vector space over $GF(q)$, and the image of a projective space $PG(r-1, q^2)$ is the projective space $PG(2r-1, q)$.

m-regular systems arising from field reduction

Group embedding

Theorem

In odd characteristic, via field reduction we get:

1. *from an Hermitian variety* $H(2n-1, q^2)$ *an hyperbolic quadric* $Q^+(4n-1, q)$*;*
2. *from an Hermitian variety* $H(2n, q^2)$ *an elliptic quadric* $Q^-(4n+1, q)$*.*

Moreover, we can define the following group inclusions:

1. $PGU(2n, q) \leq PGO^+(4n, q)$*;*
2. $PGU(2n+1, q) \leq PGO^-(4n+2, q)$*.*

m-regular systems arising from field reduction

Group embedding

The image under the *field reduction* map ϕ of an unitary transformation $M = (a_{ij}) \in GU(2n, q)$, is represented by the matrix

$$\overline{M} = (A_{ij}) = \begin{pmatrix} b_{ij} & \alpha c_{ij} \\ c_{ij} & b_{ij} + c_{ij} \end{pmatrix} \in \phi[GU(2n, q)] \leq GO^{+}(4n, q),$$

where $a_{ij} = b_{ij} + \omega c_{ij}$ and $\omega^2 = \omega + \alpha$, $\alpha \in GF(q)$.

m-regular systems arising from field reduction

Orbits on generators of hyperbolic quadrics

Theorem

The group $\phi[PGU(2n,q)] \leq PGO^+(4n,q)$ has $n+1$ orbits, say $O_{n,i}$, $0 \leq i \leq n$, on generators of $Q^+(4n-1,q)$, where

$$|O_{n,0}| = q^{n^2-n}\prod_{j=1}^{n}(q^{2j-1}+1), \quad |O_{n,n}| = \prod_{j=1}^{n}(q^{2j-1}+1),$$

$$|O_{n,i}| = q^{(n-i)(n-i-1)}\frac{\prod_{j=n-i+1}^{n}(q^{2j}-1)}{\prod_{j=1}^{i}(q^{2j}-1)}\prod_{j=1}^{n}(q^{2j-1}+1),$$

$$1 \leq i \leq n-1.$$

m-regular systems arising from field reduction

Orbits on generators of elliptic quadrics

Theorem

The group $\phi[PGU(2n+1,q^2)] \leq PGO^-(4n+2,q)$ has $n+1$ orbits, say $\widetilde{O_{n,i}}$, $0 \leq i \leq n$, on generators of $Q^-(4n+1,q)$, where

$$|O_{n,0}| = q^{n^2+n}\prod_{j=2}^{n+1}(q^{2j-1}+1), \quad |O_{n,n}| = \prod_{j=2}^{n+1}(q^{2j-1}+1),$$

$$|O_{n,i}| = q^{(n-i)(n-i+1)}\frac{\prod_{j=n-i+1}^{n}(q^{2j}-1)}{\prod_{j=1}^{i}(q^{2j}-1)}\prod_{j=2}^{n+1}(q^{2j-1}+1),$$

$$1 \leq i \leq n-1.$$

m-regular systems arising from field reduction

Theorem

If G is a group of collineations of $\mathcal{P}$ acting transitively on points of $\mathcal{P}$ and O is an orbit on the generators of $\mathcal{P}$ under the action of G, then through each point of $\mathcal{P}$ there will be a constant number of elements of O, i.e., O is a regular system of $\mathcal{P}$.

Corollary

Each one of the $n+1$ orbits $O_{n,i}$, $0 \leq i \leq n$, of $Q^+(4n-1,q)$; and each one of the $n+1$ orbits $\widetilde{O_{n,i}}$, $0 \leq i \leq n$, of $Q^-(4n+1,q)$, is a regular system of the related quadric.

Hemisystems of elliptic quadrics

We now provide a costruction of hemisystems of the elliptic quadrics $Q^-(2n+1,q)$, q odd, by partitioning the generators into generators of an hyperbolic section $Q^+(2n-1,q)$.

Proposition

Let $\mathcal{L}$ be a set of $\frac{(q^n+1)(q^{n+1}+1)}{2(q+1)}$ lines external to $Q^-(2n+1,q)$ such that

$$|\langle r, r'\rangle \cap Q^-(2n+1,q)| \neq \begin{cases} 1 & \text{if } \ |r\cap r'| = 1, \\ q+1 & \text{if } \ |r \cap r'| = 0, \end{cases}$$

for each $r, r' \in \mathcal{L}$, $r \neq r'$. Then there exists a partition of the generators of $Q^-(2n+1,q)$ into generators of a $Q^+(2n-1,q)$.

Hemisystems of elliptic quadrics

Theorem

Let $\mathcal{P}$ be a partition of the generators of the elliptic quadric $Q^-(2n+1,q)$, $n \geq 2$, into generators of hyperbolic quadrics $Q^+(2n-1,q)$ embedded in $Q^-(2n+1,q)$. Then q is odd and $2^{\frac{(q^n+1)(q^{n+1}+1)}{2(q+1)}}$ hemisystems of $Q^-(2n+1,q)$ arise, by taking one family from each of the Latin and Greek pairs in $\mathcal{P}$, and forming the union of these generators.

m-regular systems arising from k-systems

k-systems

Definition

A k-system of a polar space $\mathcal{P}$ of rank d, $1 \leq k \leq d-2$, is a set of k-spaces Π_i such that no generator containing Π_j has point in common with $\bigcup_{i \neq j} \Pi_i$.

Let $\mathcal{S}$ be a k-system of $\mathcal{P}_{d,e}$ and let $\mathcal{G}$ be the set of generators of $\mathcal{P}_{d,e}$ containing an element of $\mathcal{S}$.

Lemma

The set $\mathcal{G}$ is a $|\mathcal{M}_{\mathcal{P}_{d-k-1,e}}|$-regular system of $\mathcal{P}_{d,e}$.

m-regular systems arising from k-systems

Construction on $Q(6,3)$

Let be $Q(6,3)$ the parabolic quadric of equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 = 0. \qquad (1)$$

From the previous theorem, finding a 1-system of $Q(6,3)$ we get also a regular system of the quadric.
The set of the 7 internal points $\{P_1, P_2, \ldots, P_7\} =$

$$= \{(1:0:\ldots:0), (0:1:\ldots:0), \ldots, (0:0:\ldots:1)\}$$

is a self-polar simplex, i.e. $\forall j : P_j = \{P_i | i \neq j\}^\perp$.

m-regular systems arising from k-systems

Construction on $Q(6,3)$

Construction

Let $\pi = \langle P_1, P_2, P_3 \rangle$*, and consider the following lines:*
$r_1 = \langle P_1, P_2 \rangle$, $r_2 = \langle P_2, P_3 \rangle$, $r_3 = \langle P_1, P_3 \rangle$,
$l_1 = \langle P_4, P_5 \rangle$, $l'_1 = \langle P_6, P_7 \rangle$, $l_2 = \langle P_4, P_7 \rangle$,
$l'_2 = \langle P_5, P_6 \rangle$, $l_3 = \langle P_4, P_6 \rangle$, $l'_3 = \langle P_5, P_7 \rangle$.
Let φ *be a permutation of* $\{1, 2, 3\}$.
Let $\mathcal{R}_i$ *be one of the two reguli of the hyperbolic quadric* $\langle r_i, l_{\varphi(i)} \rangle \cap Q(6,3)$,
$\mathcal{R}'_i$ *be one of the two reguli of the hyperbolic quadric* $\langle r_i, l'_{\varphi(i)} \rangle \cap Q(6,3)$ *and*
$\mathcal{R}$ *be one of the two reguli of the hyperbolic quadric* $\pi^\perp \cap Q(6,3)$.

m-regular systems arising from k-systems

Construction on $Q(6,3)$

Proposition

The set $\mathcal{S} = \mathcal{R} \cup \left(\bigcup_{i=1}^{3} (\mathcal{R}_i \cup \mathcal{R}'_i) \right)$ is a 1-system of the quadric $Q(6,3)$.

Then the set of generators containing one line of $\mathcal{S}$ is a 4-regular system of $Q(6,3)$.

Let $\mathcal{S}^0 = \mathcal{R}^0 \cup \left(\bigcup_{i=1}^{3} (\mathcal{R}_i^0 \cup \mathcal{R}_i'^0) \right)$ be the 1-system obtained using the opposite regulus $\mathcal{R}^0$, $\mathcal{R}_i^0$ and $\mathcal{R}_i'^0$ of $\mathcal{R}$, $\mathcal{R}_i$ and $\mathcal{R}'_i$, respectively. Then the set of generators containing one line of $\mathcal{S} \cup \mathcal{S}^0$ is an 8-regular system of $Q(6,3)$.

m-regular systems w.r.t. $(k-1)$-spaces of $\mathcal{P}_{d,e}$

Association schemes

Definition

Considered a set finite set A a (symmetric) association scheme is a partition of the Cartesian product $A \times A$ into $d+1$ associate classes $C_0, C_1, \ldots, C_d$ such that:

1. $C_0 = Diag(A) = \{(\alpha, \alpha) | \alpha \in A\}$;
2. *for all i in $\{0, 1, \ldots, d\}$, C_i is symmetric, i.e. $(\alpha, \beta) \in C_i$ if and only if $(\beta, \alpha) \in C_i$;*
3. *for all i, j, k in $\{0, 1, \ldots, d\}$ there exists an integer p_{ij}^k such that, for all $(\alpha, \beta) \in C_k$:*
   $|\{\gamma \in A | (\alpha, \gamma) \in C_i \wedge (\gamma, \beta) \in C_j\}| = p_{ij}^k$.

m-regular systems w.r.t. $(k-1)$-spaces of $\mathcal{P}_{d,e}$

Distance regular graphs

Consider a graph $G = (V(G), E(G))$. Let
$G_i = \{(x, y) \in (V(G) \times V(G)) | d(x, y) = i\}$.

Definition

A graph is called distance regular if, for any two vertices v and w, the number of vertices u at distance j from u and distance k from w depends only to j, k and $i = d(v, w)$.

Definition

A graph is distance regular if $G_0, G_1, \ldots, G_d$ form an association scheme on $V(G)$.

m-regular systems w.r.t. $(k-1)$-spaces of $\mathcal{P}_{d,e}$

Dual polar graph

Definition

- *The dual polar graph $\mathcal{D}_{\mathcal{P}_{d,e}}$ of $\mathcal{P}_{d,e}$, is the graph that has as vertex set $\mathcal{M}_{\mathcal{P}_{d,e}}$, and in which two vertices x and y are adjacent if $x \cap y$ is a $(d-2)$-space of $\mathcal{P}_{d,e}$.*
- *The i-th distance graph $\mathcal{D}^i_{\mathcal{P}_{d,e}}$ of $\mathcal{P}_{d,e}$, is the graph that has as vertex set $\mathcal{M}_{\mathcal{P}_{d,e}}$, and in which two vertices x and y are adjacent if $x \cap y$ is a $(d-1-i)$-space of $\mathcal{P}_{d,e}$.*

Definition

An m-regular system w.r.t. $(k-1)$-spaces on a polar space $\mathcal{P}_{d,e}$ of rank d is a set $\mathcal{R}$ of generators such that every $(k-1)$-space of $\mathcal{P}_{d,e}$ lies on exactly m generators in $\mathcal{R}$, $0 \leq m \leq |\mathcal{M}_{\mathcal{P}_{d-k,e}}|$.

m-regular systems w.r.t. $(k-1)$-spaces of $\mathcal{P}_{d,e}$

Eigenvalues of the dual polar graph

Theorem (F. Vanhove)

$\mathcal{D}^{i}_{\mathcal{P}_{d,e}}$ has the following $d+1$ eigenvalues, $0 \leq j \leq d$:

$$\sum_{max(0,j-i) \leq u \leq min(d-i,j)} (-1)^{j+u} \begin{bmatrix} d-j \\ d-i-u \end{bmatrix}_q \begin{bmatrix} j \\ u \end{bmatrix}_q q^{\frac{(u+i-j)(u+i-j+2e-1)}{2} + \frac{(j-u)(j-u-1)}{2}}. \quad (2)$$

m-regular systems w.r.t. $(k-1)$-spaces of $\mathcal{P}_{d,e}$

Hoffman's ratio bound

Theorem (**Hoffman's ratio bound**)

Let G be a k-regular graph with vertex set $V(G)$, largest and smallest eigenvalues k and λ, respectively, and independence number $\alpha(G)$. Then

$$\alpha(G) \leq -\frac{|V(G)|\lambda}{k-\lambda}. \tag{3}$$

Corollary

$\alpha(\mathcal{D}^i_{\mathcal{P}_{d,e}}) \leq -\frac{|\mathcal{M}_{\mathcal{P}_{d,e}}|\lambda_i}{k_i - \lambda_i}$, *$k_i$ and λ_i from Equation (2).*

m-regular systems w.r.t. $(k-1)$-spaces of $\mathcal{P}_{d,e}$

Non-existence results for 1-regular systems

We study the cases when $\mathcal{R}$ is a 1-regular system of a polar space with rank 4 or 5.

Theorem

The polar spaces $Q^+(7,q)$, $H(7,q)$, $W(7,q)$, $Q(8,q)$, $H(8,q)$, $Q^-(9,q)$ do not have a 1-regular system w.r.t. lines. The polar spaces $Q^+(9,q)$, $H(9,q)$, $W(9,q)$, $Q(10,q)$, $H(10,q)$, $Q^-(11,q)$ do not have a 1-regular system w.r.t. planes.

Problem

Find the smallest eigienvalue of $\mathcal{D}^i_{\mathcal{P}_{d,e}}$.


The End