The number of Hamiltonian paths in a digraph

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A **digraph** X is a triple $X = (V, E, <)$, where $(V, <)$ is a finite linearly ordered set and E is a collection $E \subset \{(u, v) \in V \times V |\ u \neq v\}$.
For a permutation of vertices $\sigma \in \mathbb{S}_V$, denoted as a list $\sigma = (\sigma_1, \ldots, \sigma_n)$, define the X-**descent set** as

$$X\mathrm{Des}(\sigma) = \{1 \leq i \leq n-1 |\ (\sigma_i, \sigma_{i+1}) \in X\}.$$

(If V is the set $[n] = \{1, \ldots, n\}$ and $X = \{(i,j) |\ 1 \leq j < i \leq n\}$ then X-descent sets are standard descent sets of permutations $\sigma \in \mathbb{S}_n$.)

To a digraph X Stanley associated a **generating function for X-descent sets**

$$U_X = \sum_{\sigma \in \mathbb{S}_V} F_{XDes(\sigma)}. \qquad (1)$$

(R. Stanley, The X-Descent set of a permutation, Combinatorial and Algebraic Enumeration, Waterloo, Ontario, 2022, available at: https://math.mit.edu/~rstan/transparencies/gj.pdf, or R. Stanley, The X-Descent set of a permutation, Algorithmic and Enumerative Combinatorics conference, Vienna, July 4-8, 2022)

It is given in terms of **fundamental quasisymmetric functions**

$$F_I = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ for each } j \in I}} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad I \subset [n-1]. \qquad (2)$$

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Example: $n = 3$, $X = \{(1,3),(2,1),(3,1),(3,2)\}$

σ	$X\mathrm{Des}(\sigma)$
123	$\emptyset$
132	$\{1,2\}$
213	$\{1,2\}$
231	$\{2\}$
312	$\{1\}$
321	$\{1,2\}$

$$U_X = F_\emptyset + F_1 + F_2 + 3F_{\{1,2\}} = \sum_{1\le i_1\le i_2\le i_3} x_{i_1}x_{i_2}x_{i_3} + \sum_{1\le i_1< i_2\le i_3} x_{i_1}x_{i_2}x_{i_3} +$$

$$+ \sum_{1\le i_1\le i_2< i_3} x_{i_1}x_{i_2}x_{i_3} + 3\sum_{1\le i_1< i_2< i_3} x_{i_1}x_{i_2}x_{i_3}$$

Symmetric functions

Symmetric function $f = f(x_1, x_2, \dots)$ is a power series of bounded degree, invariant under any permutation of the x_i's

Partition of n is $\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, $\sum \lambda_i = n$

A partition $\lambda \vdash n$ defines the monomial symmetric function

$$\mathbf{m}_\lambda = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_j}^{\lambda_j}.$$

Other important bases for the space *Sym* of symmetric functions are elementary, homogeneous and power sum symmetric functions.

Power sums: $\mathbf{p}_k = m_{(k)} = \sum x_i^k$, $(p_0 = 1)$,

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

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U_X from our example, in the power sum basis

$$U_X = F_\emptyset + F_1 + F_2 + 3F_{\{1,2\}} = p_1^3 - p_2 p_1 + p_3$$

It holds: U_X is a $p-$integral symmetric function, i.e., $U_X = \sum_\lambda c_\lambda p_\lambda$, where $c_\lambda \in \mathbb{Z}$

Also, if ω is the linear transformation on symmetric functions given by $\omega(p_\lambda) = (-1)^{n-l(\lambda)} p_\lambda$, where $l(\lambda) = \#\{i : \lambda_i > 0\}$, then

$$\omega(U_X) = U_{\overline{X}},$$

where $\overline{X}$ is the complement of the digraph X

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Connection with Hamiltonian paths

A **Hamiltonian path** in the digraph X is a permutation $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{S}_n$ such that $(\sigma_i, \sigma_{i+1}) \in X$ for $1 \leq i \leq n-1$. Define

$$\mathbf{ham}(X) = \#\text{Hamiltonian paths in } X$$

NOTE:

$\sigma \in \mathbb{S}_n$ is a Hamiltonian path in X if and only if $X\mathrm{Des}(\sigma) = [n-1]$

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Stanley and Grinberg proved: If $U_X = \sum_\lambda c_\lambda p_\lambda$, then

$$ham(\overline{X}) = \sum_\lambda c_\lambda$$

When we apply the involution ω to $U_X = \sum_\lambda c_\lambda p_\lambda$, we obtain

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Berge's theorem

Theorem (Claude Berge) $ham(X) \equiv ham(\overline{X})\ (mod 2)$

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We construct a structure of **combinatorial Hopf algebra** on digraphs for which the enumerator U_X is obtained from a **universal morphism to quasisymmetric functions**.

Basic from combinatorial Hopf algebras

The **theory of combinatorial Hopf algebras** is founded in M. Aguiar, N. Bergeron, F. Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, Compositio Math. (2006)

A **combinatorial Hopf algebra** $(\mathcal{H}, \zeta)$ (**CHA** for short) over a field $\mathbf{k}$ is a graded, connected Hopf algebra $\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}_n$ over $\mathbf{k}$ together with a multiplicative functional $\zeta : \mathcal{H} \to \mathbf{k}$ called the **character**.

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A **partition** $\{V_1, \ldots, V_k\} \vdash V$ of the length k of a finite set V is a family of disjoint nonempty subsets with $V_1 \cup \ldots \cup V_k = V$. A **composition** $(V_1, \ldots, V_k) \models V$ is an ordered partition.

A **composition** $\alpha \models n$ is a sequence $\alpha = (a_1, \ldots, a_k)$ of positive integers with $a_1 + \cdots + a_k = n$.
The **type of a composition** $(V_1, \ldots, V_k) \models V$ is the composition $\mathrm{type}(V_1, \ldots, V_k) = (|V_1|, \ldots, |V_k|) \models n$.

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There is a bijection between sets $\mathrm{Comp}(n)$ of compositions of n and $2^{[n-1]}$ of subsets of $[n-1] = \{1, \ldots, n-1\}$ given by

$$(a_1, \ldots, a_k) \mapsto \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{k-1}\}$$

We denote the inverse of this bijection by $I \mapsto \mathrm{comp}(I)$.

The **terminal object** in the category of CHA's is the **CHA of quasisymmetric functions** $(QSym, \zeta_Q)$. A composition $\alpha = (a_1, \ldots, a_k) \models n$ defines the **monomial quasisymmetric function**

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}.$$

Alternatively we write $M_I = M_{\mathrm{comp}(I)}, \; I \subset [n-1]$.
Also, there is a basis of **fundamental quasisymmetric functions** (2) which are expressed in the monomial basis as

$$F_I = \sum_{I \subset J} M_J, \; I \subset [n-1].$$

The **character** ζ_Q is defined by $\zeta_Q(M_\alpha) = 1$ if $\alpha = (n)$ or $\alpha = ()$ and $\zeta_Q(M_\alpha) = 0$ otherwise.

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The **unique canonical morphism** $\Psi : (\mathcal{H}, \zeta) \to (QSym, \zeta_Q)$ is given on homogeneous elements with

$$\Psi(h) = \sum_{I \subset [n-1]} \zeta_I(h) M_I, \ \ h \in \mathcal{H}_n, \tag{3}$$

where ζ_I is the convolution product

$$\zeta_I = \zeta_{a_1} \cdots \zeta_{a_k} : \mathcal{H} \stackrel{\Delta^{(k-1)}}{\longrightarrow} \mathcal{H}^{\otimes k} \stackrel{proj}{\longrightarrow} \mathcal{H}_{a_1} \otimes \cdots \otimes \mathcal{H}_{a_k} \stackrel{\zeta^{\otimes k}}{\longrightarrow} \mathbf{k}$$

for $\mathrm{comp}(I) = (a_1, \ldots, a_k)$.

The **algebra of symmetric functions** *Sym* is the subalgebra of *QSym* and it is the **terminal object** in the category of **cocommutative** combinatorial Hopf algebras.

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Hopf algebra of digraphs

We say that two digraphs $X = (V, E, <)$ and $Y = (V', E', <')$ are **isomorphic** if there is an order preserving bijection $f : (V, <) \to (V', <')$ such that $(u, v) \in E$ if and only if $(f(u), f(v)) \in E'$. Any **isomorphism class of digraphs** $[X]$ has the canonical representative on the vertex set $[n] = \{1 < 2 < \cdots < n\}$ for some integer $n \geq 0$.

For digraphs $X = (V, E, <)$ and $Y = (V', E', <')$ we define the **product** $X \cdot Y$ as the digraph on the linear sum $V \oplus V' = (V \sqcup V', \prec)$ with the set of directed edges $E \cup E' \cup \{(u, v) | \ u \in V, v \in V'\}$.
The order $\prec$ on the disjoint union $V \sqcup V'$ is defined by $u \prec v$ if and only if either $u < v$ in V, $u <' v$ in V' or $u \in V, v \in V'$.

The **multiplication of digraphs** is obviously an **associative**, but **not a commutative operation**.

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Let $\mathcal{D} = \oplus_{n \geq 0} \mathcal{D}_n$ be the **graded vector space** over the field of rational numbers $\mathbb{Q}$, which is **linearly spanned by the set of all isomorphism classes of digraphs**, where the grading is given by the number of vertices. The linear extension of the product on digraphs determines the multiplication $\mu : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$, which turns the space $\mathcal{D}$ into a **noncommutative algebra**.

The **restriction of a digraph** $X = (V, E, <)$ on a subposet $S \subset V$ is the digraph $X|_S = (S, E|_S, <)$, where $E|_S = \{(u, v) \in E| \ u, v \in S\}$.

We use the restrictions of digraphs to define a **comultiplication** $\Delta : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ by

$$\Delta([X]) = \sum_{S \subset V} [X|_S] \otimes [X|_{V \setminus S}],$$

where V is the vertex set of a digraph X.

Evidently, this is a **coassociative** and a **cocommutative** operation.

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It is easy to check that Δ is an algebra morphism, meaning that

$$\Delta([X \cdot Y]) = \Delta([X]) \cdot \Delta([Y])$$

for any isomorphism classes $[X]$ and $[Y]$ of digraphs.

With these operations the space of digraphs $\mathcal{D}$ is endowed with a structure of a **graded, connected, noncommutative and cocommutative Hopf algebra**. The unit element is the digraph $\emptyset$ on the empty set of vertices and the counit is given by $\epsilon(\emptyset) = 1$ and $\epsilon([X]) = 0$ otherwise.

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To obtain a structure of a combinatorial Hopf algebra on digraphs we need a **character**. Let $\zeta : \mathcal{D} \to \mathbb{Q}$ be a linear functional determined by the following enumerator on digraphs

$$\zeta([X]) = \#\{\sigma \in \mathbb{S}_V \mid X\mathrm{Des}(\sigma) = \emptyset\}.$$

SO, $\zeta([X]$ IS EXACTLY THE NUMBER OF HAMILTONIAN PATHS IN THE COMPLEMENT OF THE DIGRAPH X

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Theorem

The linear functional ζ is multiplicative on the Hopf algebra of digraphs $\mathcal{D}$.

Proof.

Let X and Y be digraphs on V and V' respectively. For permutations $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{S}_V$ and $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{S}_{V'}$ define their reverse concatenation by $\omega \lrcorner \tau = (\tau_1, \ldots, \tau_n, \omega_1, \ldots, \omega_m) \in \mathbb{S}_{V \oplus V'}$. Then if $X\mathrm{Des}(\omega) = \emptyset$ and $Y\mathrm{Des}(\tau) = \emptyset$ it is clear that $XY\mathrm{Des}(\omega \lrcorner \tau) = \emptyset$. On the other hand any permutation $\sigma = (\sigma_1, \ldots, \sigma_{m+n}) \in \mathbb{S}_{V \oplus V'}$ with $XY\mathrm{Des}(\sigma) = \emptyset$ decomposes uniquely as $\sigma = \omega \lrcorner \tau$, where $\omega = (\sigma_{n+1}, \ldots, \sigma_{n+m}) \in \mathbb{S}_V$ and $\tau = (\sigma_1, \ldots, \sigma_n) \in \mathbb{S}_{V'}$. This shows that a map $f(\omega, \tau) = \omega \lrcorner \tau$ is a bijection, consequently

$$\zeta([XY]) = \zeta([X])\zeta([Y]).$$

□

$(\mathcal{D},\zeta)$ - CHA on digraphs

The universal morphism $\Psi : \mathcal{D} \to QSym$ assigns **to each digraph** X **a quasisymmetric function** $\Psi([X])$ determined by (3). If X is a digraph on an n-element vertex set V the coefficients of this quasisymmetric function in the monomial basis are given with

$$\zeta_I([X]) = \sum_{\substack{(V_1,\dots,V_k)\models V \\ \mathrm{type}(V_1\dots,V_k)=\mathrm{comp}(I)}} \zeta([X|_{V_1}])\cdots\zeta([X|_{V_k}]).$$

Since $\mathcal{D}$ is a cocommutative CHA, $\Psi([X])$ is a symmetric function. This implies that values of coefficients ζ_I depend only on set partitions of the vertex set V.

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A generating function for X-descent sets

The expansion of the enumerator U_X defined by (1) in the basis of monomial quasisymmetric functions is given by

$$U_X = \sum_{\sigma\in\mathbb{S}_V} \sum_{XDes(\sigma)\subset I} M_I = \sum_{I\subset[n-1]} \mu_I(X) M_I, \qquad (4)$$

where n is the number of vertices and coefficients are determined with

$$\mu_I(X) = \#\{\sigma \in \mathbb{S}_V | \ XDes(\sigma) \subset I\}.$$

Theorem

Let $\Psi : \mathcal{D} \to QSym$ be a universal morphism from the combinatorial Hopf algebra of digraphs to quasisymmetric functions. Then for a digraph X

$$\Psi([X]) = U_X.$$

Proof.

According to expansions in the basis of monomial quasisymmetric functions we should prove that $\zeta_I([X]) = \mu_I(X)$ for all $I \subset [n-1]$. For a subset $I \subset [n-1]$ let $\sigma \in \mathbb{S}_n$ be a permutation such that $XDes(\sigma) \subset I$. If $\mathrm{comp}(I) = (a_1, \ldots, a_k)$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$ let $V_1 = \{\sigma_1, \ldots, \sigma_{a_1}\}, V_2 = \{\sigma_{a_1+1}, \ldots, \sigma_{a_1+a_2}\}, \ldots, V_k = \{\sigma_{a_1+\cdots+a_{k-1}+1}, \ldots, \sigma_n\}$ be linearly ordered subposets of $[n]$. Then $(\sigma_1, \ldots, \sigma_{a_1}) \in \mathbb{S}_{V_1}, (\sigma_{a_1+1}, \ldots, \sigma_{a_1+a_2}) \in \mathbb{S}_{V_2}, \ldots, (\sigma_{a_1+\cdots+a_{k-1+1}}, \ldots, \sigma_n) \in \mathbb{S}_{V_k}$.

In this way the permutation σ produces a composition $(V_1, \ldots, V_k) \models [n]$ with $\mathrm{type}(V_1, \ldots, V_k) = \mathrm{comp}(I)$ which satisfies $X|_{V_1} Des(\sigma_1, \ldots, \sigma_{a_1}) = \emptyset, \ldots, X|_{V_k} Des(\sigma_{a_1+\cdots+a_{k-1}+1}, \ldots, \sigma_n) = \emptyset$.

In other direction, for any composition $(V_1, \ldots, V_k) \models [n]$ with $\mathrm{type}(V_1, \ldots, V_k) = \mathrm{comp}(I)$, the concatenation of permutations $\sigma_1 \in \mathbb{S}_{V_1}, \ldots, \sigma_k \in \mathbb{S}_{V_k}$ such that $X|_{V_1} Des(\sigma_1) = \emptyset, \ldots, X|_{V_k} Des(\sigma_k) = \emptyset$ gives the permutation

Tournaments

A **tournament** is a digraph $X = (V, E, <)$ such that for all vertices $u, v \in V$ either $(u, v) \in E$ or $(v, u) \in E$

Tournaments are obviously closed under operations of taking restrictions and products, so their isomorphism classes generate a Hopf subalgebra $\mathcal{T} \subset \mathcal{D}$ of the Hopf algebra of digraphs

$\mathcal{T}$ - **the CHA of tournaments**

Each tournament $X = (V, E, <)$ has its complementary tournament $\overline{X} = (V, \overline{E}, <)$ determined by $(u, v) \in E$ if and only if $(v, u) \in \overline{E}$

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Theorem

For each tournament we have $U_X = U_{\overline{X}}$.

Proof.

Assume that X is a tournament on the vertex set $[n]$. Define reversions of a permutation $\sigma = (i_1, \ldots, i_n) \in \mathbb{S}_n$ and of a subset $I \subset [n-1]$ by $\mathrm{rev}\sigma = (i_n, \ldots, i_1)$ and $\mathrm{rev} I = \{n - i \mid i \in I\}$. For a permutation $\sigma \in \mathbb{S}_n$ we have

$$\mathrm{rev} XDes(\sigma) = \overline{X}Des(\mathrm{rev}\sigma),$$

so $XDes(\sigma) \subset I$ if and only if $\overline{X}Des(\mathrm{rev}\sigma) \subset \mathrm{rev} I$ for any subset $I \subset [n-1]$. Therefore $\mu_I(X) = \mu_{\mathrm{rev} I}(\overline{X}), I \subset [n-1]$. The operation of reversion can be defined on the monomial bases by $\mathrm{rev} M_I = M_{\mathrm{rev} I}$. Using the expansion (4) we obtain

$$U_X = \mathrm{rev}\, U_{\overline{X}}.$$

The statement follows since the reversion on symmetric functions is an identity. □

THANK YOU FOR YOUR ATTENTION!

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