Dual incidences arising from a subsets of spaces*

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Motivation and definitions

A design with parameters $t-(v,k,\lambda)$ is a collection $\mathcal{B}$ of k-element subsets (blocks) of a v-element set $\mathcal{P}$, such that each t-element subset of $\mathcal{P}$ is contained in exactly λ blocks from $\mathcal{B}$.

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Definition

Let V be a n-dimensional vector space over $\mathbb{F}_q$. The pair $(V,\mathcal{H})$ is a $t-(n,k,\lambda)_q$ design if $\mathcal{H} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$ and if for every $T \in \begin{bmatrix} V \\ t \end{bmatrix}_q$ there are exactly λ subspaces $H \in \mathcal{H}$ such that $T \leq H$.

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$\mathcal{D}_{max}(\mathcal{H}) = \{\mathcal{H}_M \mid M \leq V, \ \dim M = n-1\}$ and
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especially in the case when $\mathcal{H}$ is a $t-(n,k,\lambda)_q$ design.

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$$Res_M(\mathcal{H}) = (M, \mathcal{H}_M).$$

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Let $C = (C_{ij})$ be a matrix, with the entries from the set $\{0, 1\}$, such that $C_{ij} = 1$ if $M_j \cap W_i = \{0\}$ (trivial intersection).

Duality of $\mathcal{D}_{max}(\mathcal{H})$ and $\mathcal{D}_{min}(\mathcal{H})$

Theorem (duality):

Let $\mathcal{H} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$, where A and B are incidence matrices of $\mathcal{D}_{min}(\mathcal{H})$ and $\mathcal{D}_{max}(\mathcal{H})$. Then the following holds:

- label=() $A = J - \frac{1}{q^{n-k-1}} CB$,
- label=() $B = J - \frac{1}{q^{k-1}} C^t A$.

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the following holds (in the group ring $\mathbb{Q}[V]$):

$$\mathcal{H}_{M_i} = \mathcal{H} - \frac{1}{q^{k-1}} \sum_{W_j \cap M_i = \{0\}} \mathcal{H}_{W_j}, \ \mathcal{H}_{W_j} = \mathcal{H} - \frac{1}{q^{n-k-1}} \sum_{W_j \cap M_i = \{0\}} \mathcal{H}_{M_i}.$$

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$|\mathcal{H}| = \lambda \begin{bmatrix} n \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q$, $|\mathcal{H}_{W_j}| = \lambda \begin{bmatrix} n-1 \\ t-1 \end{bmatrix}_q / \begin{bmatrix} k-1 \\ t-1 \end{bmatrix}_q$, and

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$|\mathcal{H}_{W_j} \cap \mathcal{H}_{W_s}| = \lambda \begin{bmatrix} n-2 \\ t-2 \end{bmatrix}_q / \begin{bmatrix} k-2 \\ t-2 \end{bmatrix}_q$, where $W_j \neq W_s$.

Duality and incidence matrices

Theorem:

Let $\mathcal{H}$ be a $t-(n,k,\lambda)_q$. The incidence matrix A of $\mathcal{D}_{min}(\mathcal{H})$ satisfies the following:

- label=() $AA^t = (\alpha_1 - \alpha_2)\lambda I + \alpha_2 \lambda J$,
- label=() $JA = \begin{bmatrix} k \\ 1 \end{bmatrix}_q J$.

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Lemma:

If $\mathcal{H}$ is a $t-(n,k,\lambda)_q$ design and M is a hyperplane, then $|\mathcal{H}_M| = \frac{(\alpha_0 - \alpha_1)}{q^k} \cdot \lambda$. If M and N are distinct hyperplanes, then $|\mathcal{H}_M \cap \mathcal{H}_N| = \frac{\begin{bmatrix} n-2 \\ k \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q} \cdot \lambda.$

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Theorem:

Let $\mathcal{H}$ be a $t-(n,k,\lambda)_q$ design. The incidence matrix B of $\mathcal{D}_{max}(\mathcal{H})$ satisfies the following:

- label=() $BB^t = \lambda(\alpha_0 - \beta)I + \beta\lambda J$, where $\beta = \begin{bmatrix} n-2 \\ k \end{bmatrix}_q / \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$,
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Thank you!