Relations on Nets and MOLS

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Includes joint work with Michael Gill

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00	98	49	85	73	37	16	24	61	52
96	11	84	57	29	72	60	43	08	35
64	47	22	76	81	18	39	05	50	93
59	65	78	33	06	80	41	92	27	14
38	82	07	91	44	69	75	56	13	20
83	26	90	19	67	55	02	31	74	48
25	34	51	40	12	03	88	77	99	66
42	53	15	04	30	21	97	68	86	79
17	70	36	62	58	94	23	89	45	01
71	09	63	28	95	46	54	10	32	87

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$$\begin{bmatrix} 00 & 98 & 49 & 85 & 73 & 37 & 16 & 24 & 61 & 52 \\ 96 & 11 & 84 & 57 & 29 & 72 & 60 & 43 & 08 & 35 \\ 64 & 47 & 22 & 76 & 81 & 18 & 39 & 05 & 50 & 93 \\ 59 & 65 & 78 & 33 & 06 & 80 & 41 & 92 & 27 & 14 \\ 38 & 82 & 07 & 91 & 44 & 69 & 75 & 56 & 13 & 20 \\ 83 & 26 & 90 & 19 & 67 & 55 & 02 & 31 & 74 & 48 \\ 25 & 34 & 51 & 40 & 12 & 03 & 88 & 77 & 99 & 66 \\ 42 & 53 & 15 & 04 & 30 & 21 & 97 & 68 & 86 & 79 \\ 17 & 70 & 36 & 62 & 58 & 94 & 23 & 89 & 45 & 01 \\ 71 & 09 & 63 & 28 & 95 & 46 & 54 & 10 & 32 & 87 \end{bmatrix}$$

Euler famously conjectured these shouldn't exist!

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A *k-net* has k orthogonal parallel classes (corresponds to $(k-2)$ MOLS).

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The union of an even number of parallel classes is a *trivial relation*. Any other relation is *non-trivial*.

A non-trivial relation

$\{0, 1, 2, 3\} \mapsto 1$
$\{4, \ldots, 9\} \mapsto 0$

$$\begin{bmatrix}
11 & 00 & 00 & 00 & 01 & 10 & 10 & 10 & 01 & 01 \\
00 & 11 & 00 & 00 & 10 & 01 & 01 & 01 & 10 & 10 \\
00 & 00 & 11 & 00 & 01 & 10 & 10 & 10 & 01 & 01 \\
00 & 00 & 00 & 11 & 10 & 01 & 01 & 01 & 10 & 10 \\
10 & 01 & 10 & 01 & 00 & 00 & 00 & 00 & 11 & 11 \\
01 & 10 & 01 & 10 & 00 & 00 & 11 & 11 & 00 & 00 \\
10 & 10 & 01 & 01 & 11 & 11 & 00 & 00 & 00 & 00 \\
01 & 01 & 10 & 10 & 11 & 11 & 00 & 00 & 00 & 00 \\
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00 & 00 & 00 & 11 & 10 & 01 & 01 & 01 & 10 & 10 \\
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01 & 10 & 01 & 10 & 00 & 00 & 11 & 11 & 00 & 00 \\
10 & 10 & 01 & 01 & 11 & 11 & 00 & 00 & 00 & 00 \\
01 & 01 & 10 & 10 & 11 & 11 & 00 & 00 & 00 & 00 \\
10 & 01 & 10 & 01 & 00 & 00 & 11 & 00 & 00 & 11 \\
01 & 10 & 01 & 10 & 00 & 00 & 00 & 11 & 11 & 00
\end{bmatrix}$$

The *type* of a relation lists the number of relational lines from each parallel class.
The above relation has type 4444 (also written 4^4).

Another relation of type 4444 on the same net

$\{2,3,4,5\} \mapsto 1$
$\{0,1,\ 6,7,8,9\} \mapsto 0$

$$\begin{bmatrix}
00 & 00 & 10 & 01 & 01 & 10 & 00 & 11 & 00 & 11 \\
00 & 00 & 01 & 10 & 10 & 01 & 00 & 11 & 00 & 11 \\
01 & 10 & 11 & 00 & 00 & 00 & 10 & 01 & 10 & 01 \\
10 & 01 & 00 & 11 & 00 & 00 & 10 & 01 & 10 & 01 \\
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37	$\sim 10^{15}$

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A more natural alternative(?) might be to suggest that

$$\text{rank}_p(N_k) \geqslant kn - \binom{k+1}{2}.$$

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Theorem: Affine planes must satisfy an odd relation.

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You can then write down a set of equations involving *point types*.
A point type is a binary 10-vector specifying 2 bits for each parallel class saying whether the point is or is not on a relational line for each relation.

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- [Pipe dream] Develop the theory to the point that you can rule out some projective planes.