Existence of small ordered orthogonal arrays

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(joint work with Kai-Uwe Schmidt)

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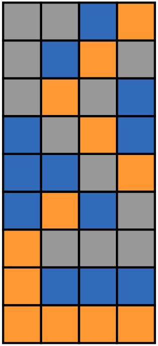
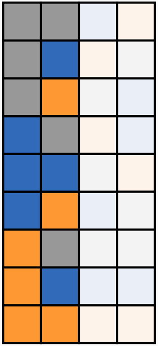
Germany

Orthogonal arrays



2-$(3, 4, 1)$ orthogonal array

Orthogonal arrays



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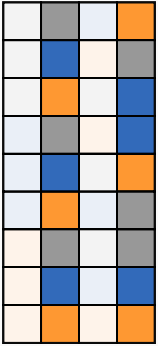

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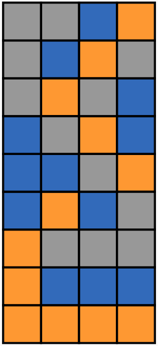

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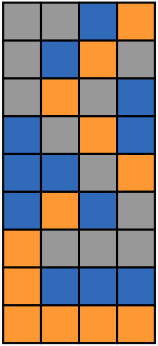

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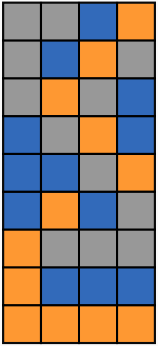

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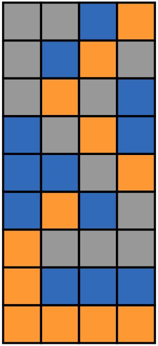

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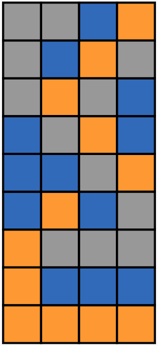

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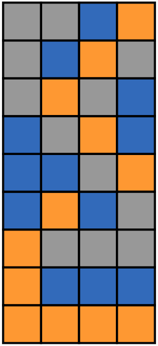

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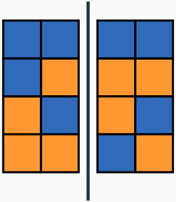
Applications:

statistics, coding theory, cryptography, software testing, ...

Ordered orthogonal array (OOA)



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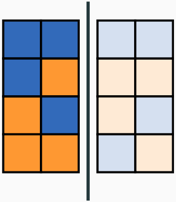

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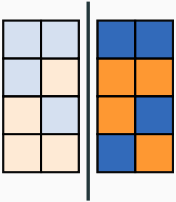

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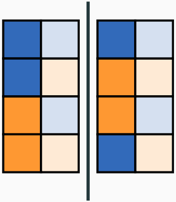

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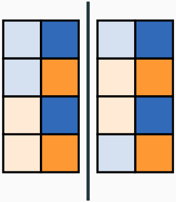

No orthogonal array
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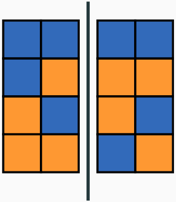

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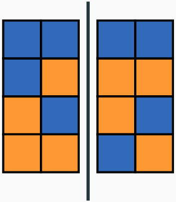

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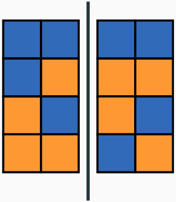

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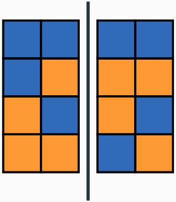

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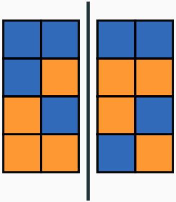

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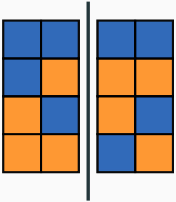

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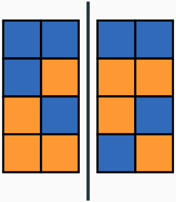

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Applications:

numerical integration (connected to (t, m, s)-nets), coding theory, cryptography, ...

Main questions

Trivial examples:

The complete set of n-tuples or nr-tuples on q symbols is a t-orthogonal array or t-OOA for all t, respectively.

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$N(n)$ = minimum number N such that a t-(q, n, λ) orthogonal array with N rows exists for some λ.

Accordingly, define $N^*(n, r)$ for t-(q, n, r, λ) OOAs.

Lower bounds

Orthogonal array: Rao bound 1973

$$N(n) \geq \left(\frac{cqn}{t}\right)^{t/2}$$

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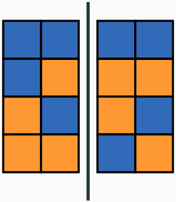

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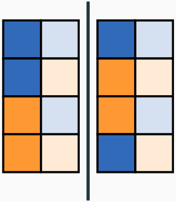

Choose only the first column in every block of the OOA.

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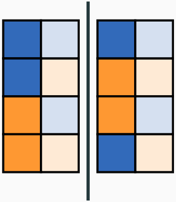

Choose only the first column in every block of the OOA.

$$N^*(n, r) \geq N(n) \geq \left(\frac{cqn}{t}\right)^{t/2}$$

Existence of orthogonal arrays

Theorem (Kuperberg-Lovett-Peled 2017)

For all integers q, n, t with $q \geq 2$ and $1 \leq t \leq n$, there exists a t-(q, n, λ) orthogonal array Y such that

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This gives

$$\left(\frac{c'qn}{t}\right)^{t/2} \leq N(n) \leq \left(\frac{cqn}{t}\right)^{ct}$$

for some universal constants $c, c' > 0$.

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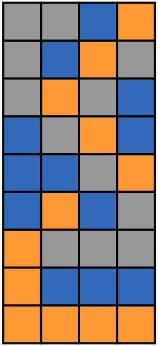

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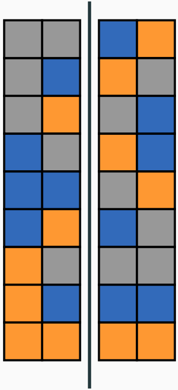

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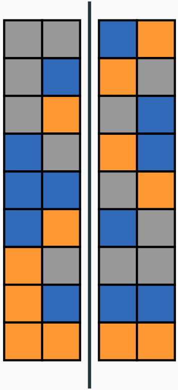

Divide the nr columns into n blocks each of size r.

$$N^*(n, r) \leq N(nr) \leq \left(\frac{cqnr}{t}\right)^{ct}$$

Main result

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Theorem (Schmidt-W. 2023)

For all integers q, n, r, t with $q \ge 2$ and $1 \le t \le nr$, there exists a t-(q, n, r, λ) ordered orthogonal array Y such that

$$|Y| \le \left(\frac{cq(n+t)}{t}\right)^{ct}$$

for some universal constant $c > 0$.

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The proof is nonconstructive and based on a probabilistic method.

Constructions of OOAs

Besides using (t, m, s)-nets, only a few constructions of OOAs are known, for example:

- Rosenbloom-Tsfasman (1997)
- Skriganov (2001)
- Castoldi-Moura-Panario-Stevens (2017)
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They all give MDS-like codes, namely optimal t-$(q, n, r, 1)$ OOAs of size q^t if q is a prime power with $q \geq n - 1$.

KLP theorem

Kuperberg, Lovett, and Peled (2017) established a theorem that proves the existence of "regular combinatorial objects" by probabilistic techniques.

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It has been applied to

- orthogonal arrays, combinatorial t-designs, t-wise permutations (Kuperberg-Lovett-Peled 2017)
- t-designs over finite fields (Fazeli-Lovett-Vardy 2014)
- large sets of combinatorial t-designs (Lovett-Rao-Vardy 2020)
- large sets of t-designs over finite fields (Bao-Ji 2022)
- ...

Basic idea of the KLP theorem

"Regular combinatorial objects": highly symmetric objects with many simultaneous conditions of exact count.

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t-(q, n, r, λ) OOA:
collection of vectors in $[q]^{nr}$ such that on any t coordinates (that are allowed to choose), each one of the possible q^t patterns occurs exactly λ times.

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Basic idea of KLP theorem

If the regular combinatorial objects satisfy certain properties, then the probability that a random construction works is positive, albeit tiny. Thus, the object exists.

Framework of KLP theorem

Let M be an integer matrix with row set R and column set C.

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Goal: Find a *small* subset Y of rows whose average equals the average of all rows

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Incidence matrix

$$
\begin{array}{cc}
 & \begin{array}{cccc} {\color{orange}\blacksquare\blacksquare}* & {\color{blue}\blacksquare}{\color{orange}\blacksquare}* & {\color{blue}\blacksquare}*{\color{orange}\blacksquare} & \cdots \end{array} \\
M = \begin{array}{c}
{\color{orange}\blacksquare\blacksquare\blacksquare} \\
{\color{blue}\blacksquare}{\color{orange}\blacksquare\blacksquare} \\
{\color{orange}\blacksquare}{\color{blue}\blacksquare}{\color{orange}\blacksquare} \\
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{\color{orange}\blacksquare}{\color{blue}\blacksquare\blacksquare} \\
{\color{blue}\blacksquare\blacksquare\blacksquare}
\end{array} &
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
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\end{array}
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$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

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1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
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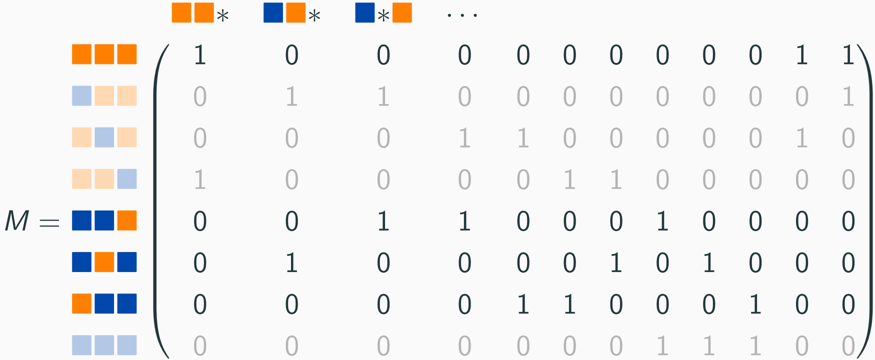
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
$$

$$
\sum_{x \in R} \mathrm{row}(x) = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}
$$

Incidence matrix



$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\sum_{x \in R} \mathrm{row}(x) = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$\sum_{x \in Y} \mathrm{row}(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Incidence matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

This gives

$$\frac{1}{4}(1,\dots,1) = \frac{1}{|Y|}\sum_{x\in Y} \mathrm{row}(x) = \frac{1}{|R|}\sum_{x\in R} \mathrm{row}(x) = \frac{1}{8}(2,\dots,2).$$

Orthogonal arrays

A subset Y of rows of M satisfying

$$\frac{1}{|Y|}\sum_{x\in Y}\mathrm{row}(x) = \frac{1}{|R|}\sum_{x\in R}\mathrm{row}(x)$$

is precisely a t-(q, n, λ) orthogonal array.

KLP theorem

Theorem (KLP theorem)
If the matrix M satisfies certain conditions, then there is a small subset Y of rows in M such that

$$\frac{1}{|Y|}\sum_{x\in Y}\mathrm{row}(x) = \frac{1}{|R|}\sum_{x\in R}\mathrm{row}(x).$$

(Small means polynomial in the number of columns of M and other parameters.)

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Symmetry:

The *symmetry group* of M acts transitively on the rows of M.

Divisibility condition

We want a *small* subset Y with

$$\sum_{x \in Y} \mathrm{row}(x) = |Y| \cdot \frac{1}{|R|} \sum_{x \in R} \mathrm{row}(x).$$

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$$\sum_{x \in Y} \mathrm{row}(x) = |Y| \cdot \frac{1}{|R|} \sum_{x \in R} \mathrm{row}(x).$$

Divisibility: There exists a *small* integer c such that

$$c \cdot \frac{1}{|R|} \sum_{x \in R} \mathrm{row}(x)$$

can be expressed as an integer combination of the rows of M.

Boundedness of $V^\perp$

Let $V^\perp$ be the orthogonal complement of V in $\mathbb{Q}^R$.

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The subspace $V^\perp$ is spanned by "short" integer vectors.

This is usually the hardest condition to check!

Main result

Theorem (Schmidt-W. 2023)

For all integers q, n, r, t with $q \geq 2$ and $1 \leq t \leq nr$, there exists a t-(q, n, r, λ) ordered orthogonal array Y such that

$$|Y| \leq \left(\frac{cq(n+t)}{t}\right)^{ct}$$

for some universal constant $c > 0$.