On the diameter and zero forcing number of some graph classes in the Johnson, Grassmann and Hamming association scheme

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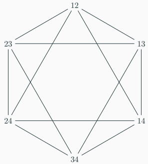
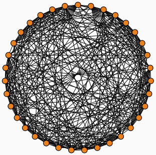
Background and motivation

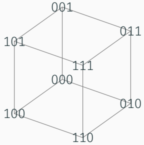
Johnson, Grassmann and Hamming graphs

	Johnson	Grassmann	Hamming
Notation	$J(n,k)$	$J_q(n,k)$	$H(n,q)$
Vertices	$\binom{[n]}{k}$	k-dim. subspaces of $\mathbb{F}_q^n$	$\{0,1,\ldots,q-1\}^n$
Edges	$\lvert u \cap v\rvert = k-1$	$\dim u \cap v = k-1$	$q-1$ entries same



$J(4,2)$


$J_2(4,2)$



$H(3,2)$

Generalized Johnson and Grassmann graphs

Let $S \subseteq \{0, 1, \dots, k-1\}$

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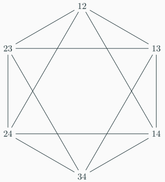
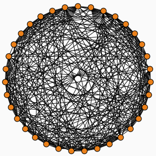
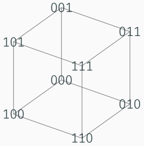


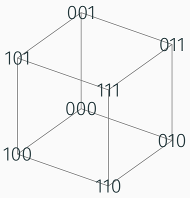




Why (generalized) Johnson, Grassmann and Hamming graphs?

Structure: distance-regular $\rightarrow$ rich algebraic structure

Applications: designs, codes, association schemes, …



Why (generalized) Johnson, Grassmann and Hamming graphs?

(Chen, Lih 1987) Hamiltonicity generalized Johnson graphs

(Van Dam, Haemers, Koolen, Spence 2006) Johnson and Grassmann graphs not determined by their spectrum

(Meagher, Bailey 2012) Metric dimension of Grassmann graphs

(Alspach 2013) Johnson graphs Hamiltonian connected

(Balogh, Cherkashin, Kiselev 2019) Coloring of generalized Kneser graphs

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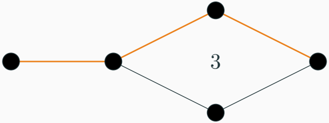
Diameter, zero forcing?

Graph diameter

Largest distance between two vertices

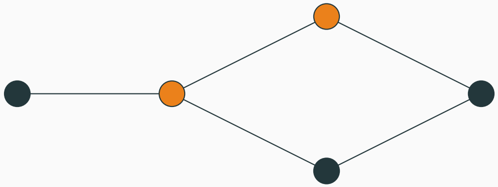


Polynomial-time computable, but

- our graphs are large;
- finding a closed expression is hard

Zero forcing on graphs

Graph $G = (V, E)$ with set $B \subseteq V$ of orange vertices



Force: unique uncolored neighbor of a orange vertex is colored orange

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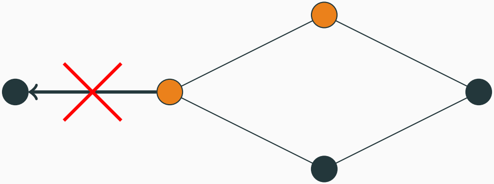

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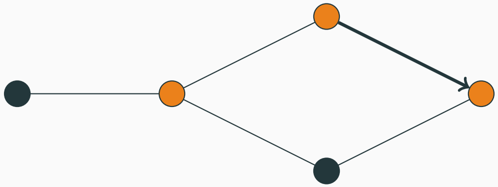

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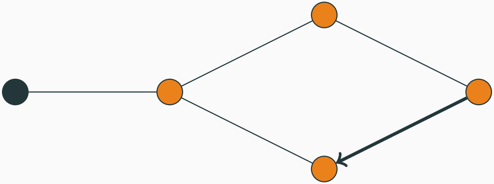

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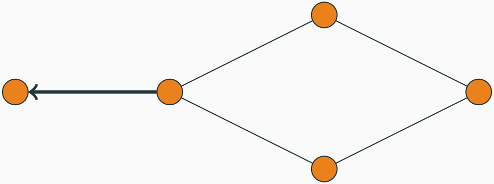

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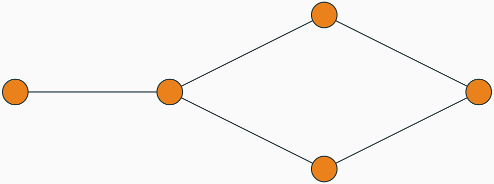


Zero forcing number $Z(G)$: minimum $|B|$ such that all of V is forced

(Yang 2013) In general, this is NP-hard

History and applications

(Haynes, Hedetniemi, Hedetniemi, Henning 2002) Power domination (placing Phasor Measurement Units in electrical networks)

(Burgarth, Giovannetti 2007) Zero forcing for quantum system control

(AIM workshop 2008) Zero forcing as an upper bound for minimum rank

$\updownarrow$?

(Alon 2008) Zero forcing on Cayley graphs, relation minimum rank

Known results

(Agong, Amarra, Caughman, Herman, Terada 2018) Diameter and girth of generalized Johnson graphs

(Fallat, Meagher, Soltani, Yang 2016) $Z(J(n,2)) = \binom{n}{2} - n + 2$

(Brešar, Gologranc, Kos 2016) $Z(K(n,k)) = \binom{n}{k} - \binom{2k}{k} \quad$ if $n \geq 3k+1$

(AIM workshop 2008) $Z(H(2,q)) = q^2 - 2q + 2.$

(AIM workshop, Alon 2008) $Z(H(n,2)) = 2^{n-1}$

Our contribution

- Diameter: generalized Grassmann graphs
- Zero forcing: Hamming graphs and generalized Johnson, Grassmann graphs

Diameter

Generalized Grassmann graphs

Let $S \subseteq \{0, 1, \ldots, k-1\}$

	Johnson	Grassmann	Hamming
Notation	$J_S(n,k)$	$J_{q,S}(n,k)$	$H(n,q)$
Vertices	$\binom{[n]}{k}$	k-dim. subspaces of $\mathbb{F}_q^n$	$\{0,1,\ldots,q-1\}^n$
Edges	$\lvert u \cap v \rvert \in S$	$\dim u \cap v \in S$	$q-1$ entries same

$$J_{q,S}(n,k) \simeq J_{q,\{s+n-2k \mid s \in S\}}(n, n-k) \quad \rightarrow \quad \text{assume } n \geq 2k$$

Diameter and girth

(Agong, Amarra, Caughman, Herman, Terada 2018) Diameter and girth of generalized Johnson graphs

(Caughman, Herman, Terada 2023) Distance function and odd girth of generalized Johnson graphs

Diameter and girth

Nice fact: #trivially intersecting k-subspaces of $\mathbb{F}_q^n$ >> #disjoint k-subsets of $[n]$.

Theorem

Let $n \geq 2k$ and $s = \min S$. Then

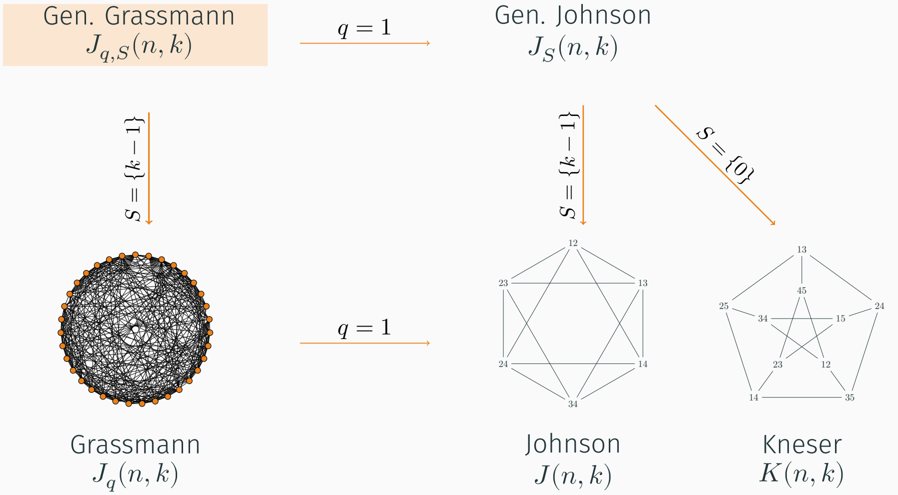
$$\operatorname{diam}\left(J_{q,S}(n,k)\right) = \begin{cases} 2 & \text{if } s = 0 \\ \left\lceil \frac{k}{k-s} \right\rceil & \text{if } s \neq 0. \end{cases}$$

Theorem

Every generalized Grassmann graph with $S \neq \emptyset$ has girth 3.

Zero forcing

Relations between families



Known results

(Fallat, Meagher, Soltani, Yang 2016)

$$Z(J(n,2)) = \binom{n}{2} - n + 2$$

(Brešar, Gologranc, Kos 2016)

$$Z(K(n,k)) = \binom{n}{k} - \binom{2k}{k}$$

if $n \geq 3k+1$; upper bound for $n \leq 3k$

The case $\max(S) = s$

Theorem

Let $S \subseteq \{0, 1, \dots, k-3\}$ with $s := \max(S)$, and $n \geq \max(3k - 2s, 2k+1)$. Then

$$Z(J_{q,S}(n,k)) \leq \begin{bmatrix} n \\ k \end{bmatrix}_q - \binom{2k-2s}{k-s}.$$

If $S = \{0, 1, \dots, s\}$, equality holds throughout.

Note: independent of n and q

Corollaries for subfamilies

gen. Grassmann $$Z(J_{q,S}(n,k)) \leq \begin{bmatrix} n \\ k \end{bmatrix}_q - \binom{2k-2s}{k-s}$$

$$\downarrow$$

gen. Johnson $$Z(J_S(n,k)) \leq \binom{n}{k} - \binom{2k-2s}{k-s}$$

$$\downarrow$$

Kneser (BGK 2016) $$Z(K(n,k)) = \binom{n}{k} - \binom{2k}{k}$$

Conditions

Theorem
Let $S \subseteq \{0, 1, \dots, k-3\}$ with $s := \max(S)$, and $n \geq \max(3k-2s, 2k+1)$. Then

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What about $(n, k, s) = (9, 4, 1)$?

Conjecture

Conjecture

Let $S \subseteq \{0, 1, \dots, k-3\}$ and $n \geq 2k+1$, where $s := \max(S)$. Then

$$Z(J_S(n,k)) \leq \begin{bmatrix} n \\ k \end{bmatrix}_q - \binom{2k-2s}{k-s}.$$

If $S = \{0, 1, \dots, s\}$, equality holds throughout.

Computational experiments for gen. Johnson graphs suggest this is true

Construction found for $s = 1$

The case $\min(S) = s$

Only for generalized Johnson graphs:

Theorem

Let $S \subseteq \{0, 1, \dots, k-1\}$ with $s := \min(S)$ and $n \geq 2k - s$. Then

$$Z(J_S(n,k)) \leq \binom{n}{k} - \binom{n - 2(k-s)}{s}.$$

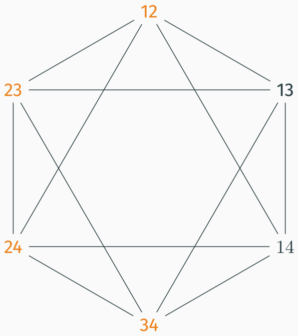
If $S = \{s, s+1, \dots, k-1\}$, equality holds throughout.

Proving zero forcing bounds

- Upper bounds are ‘easy’: find a construction
- Lower bounds are hard: maximum nullity, Grundy domination, etc.

Upper bound for Johnson graphs

$$B = V \setminus \{v \in V \mid 1 \in v,\ 2 \notin v\} \quad \rightarrow \quad Z(J(n,k)) \leq |B| = \binom{n}{k} - \binom{n-2}{k-1}$$



Force v with $(v \setminus 1) \cup 2$: $24 \rightarrow 14,\ 23 \rightarrow 13$

Upper bound for Johnson graphs

Johnson $V \setminus \{A \in V \mid \underline{1} \in A,\ \underline{2} \notin A\}$

$\downarrow$

gen. Johnson $V \setminus \{A \in V \mid \underline{[k-s]} \subset A,\ \underline{k-s+1, \ldots, 2(k-s)} \notin A\}$

Hamming graphs

Let $S \subseteq \{0, 1, \dots, k-1\}$

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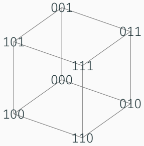


Main result

(AIM workshop 2008) $Z(H(2,q)) = q^2 - 2q + 2.$

(AIM workshop, Alon 2008) $Z(H(n,2)) = 2^{n-1}$

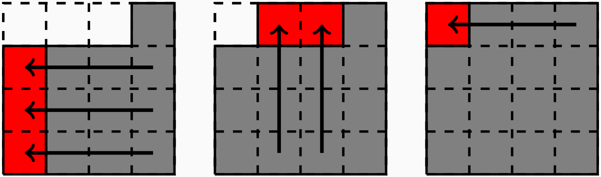
Theorem

For any $n, q \geq 2$,

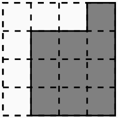
$$Z(H(n,q)) = \frac{1}{2}\left(q^n + (q-2)^n\right).$$

A constructive upper bound

If $n = 2$, the following is a zero forcing set:

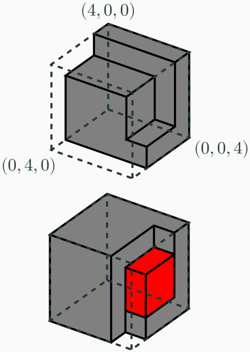


A constructive upper bound



$H(n, q) = H(n-1, q) \square K_q$

$\rightarrow$ take q copies of the zero forcing set of $H(n-1, q)$, remove core vertices from one



Closing remarks

Open problems

- Get rid of lower bound on n for generalized Grassmann graphs;

Theorem
Let $S \subseteq \{0, 1, \dots, k-3\}$ and $n \geq \max(3k - 2s, 2k + 1)$, where $s := \max(S)$. Then

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If $S = \{0, 1, \dots, s\}$, equality holds throughout.

Open problems

- Get rid of lower bound on n for generalized Grassmann graphs;
- Zero forcing number of classic Grassmann graphs;
- Zero forcing on distance-regular graphs in general.

arXiv:2302.07757