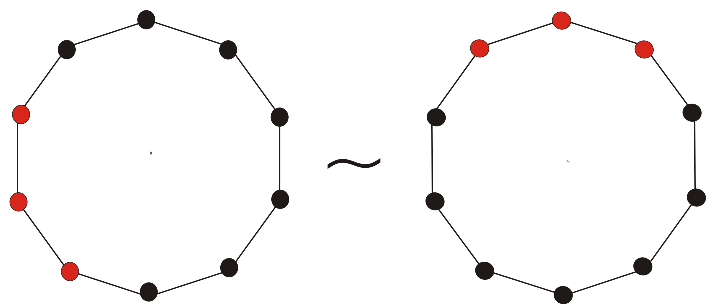
COUNTING SYMMETRIC BRACELETS

Yuliya Zelenyuk

Necklaces and Bracelets

Let $\mathbb{Z}_n$ be a finite cyclic group. We are coloring its vertices in r colours. The total number of colorings is r^n. Colorings are equivalent if we can get one from another by rotation.



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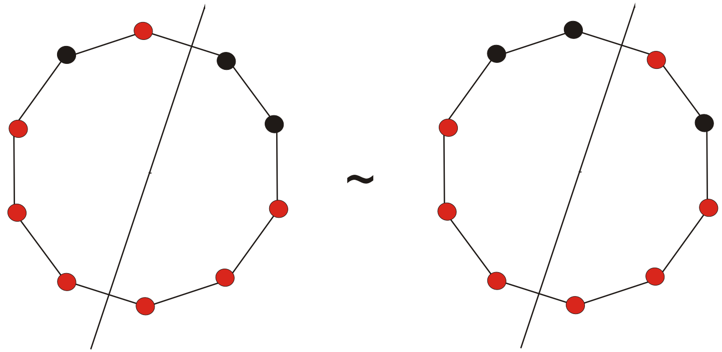

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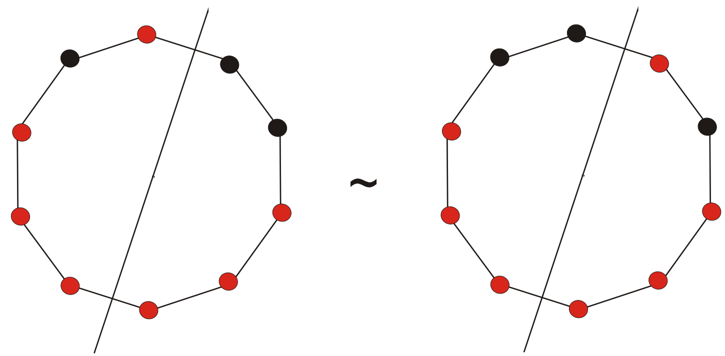

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$$ds(d) = \sum_{k|d} r^k \mu(\frac{d}{k}),$$

where $\mu(x)$ is Möbius function.

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I. VINOGRADOV *Elementary Number Theory*, 1972.

The number of r-ary necklaces of length n is

$$N_r(n) = \frac{1}{n} \sum_{d|n} \varphi(d) r^d,$$

where φ is the Euler function.

The number of r-ary bracelets of length n is

$$B_r(n) = \frac{1}{2}N_r(n) + \begin{cases} \frac{1}{4}(r+1)\,r^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \frac{1}{2}r^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

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So,

$$[\chi] = \{\chi g : g \in G\}$$

is an orbit and

$$St(\chi) = \{g \in G : \chi g = \chi\}.$$

is a stabilizer of χ.

The number of orbits (r-ary necklaces) of group G equals

$$N_r(G) = \frac{1}{|G|} \sum_{g \in G} r^{|G:\langle g \rangle|}$$

where $\langle g \rangle$ is the subgroup generated by g.
The number of r-ary bracelets of abelian group A equals

$$B_r(A) = \frac{1}{2} N_r(A) + \\ + \frac{1}{2|A[2]|} \left(r^{\frac{|A[2]|}{2}} + |A[2]| - 1 \right) r^{\frac{|A|}{2}}$$

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$$x \mapsto g x^{-1} g$$

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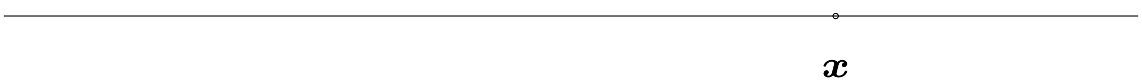

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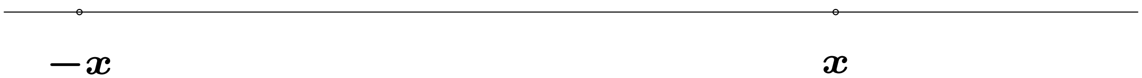

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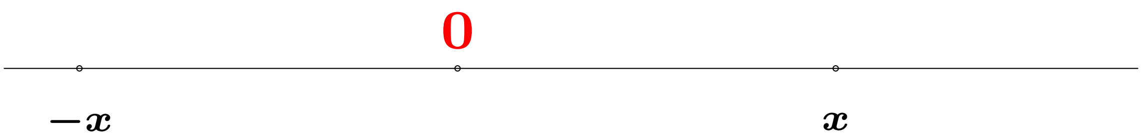

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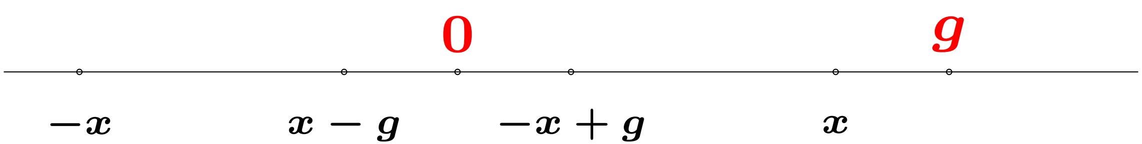

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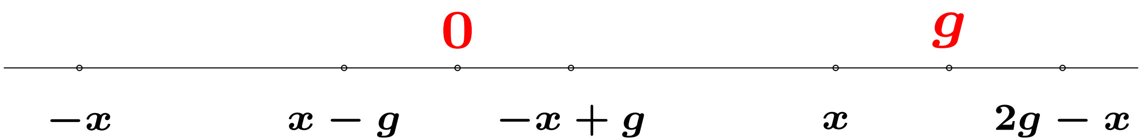

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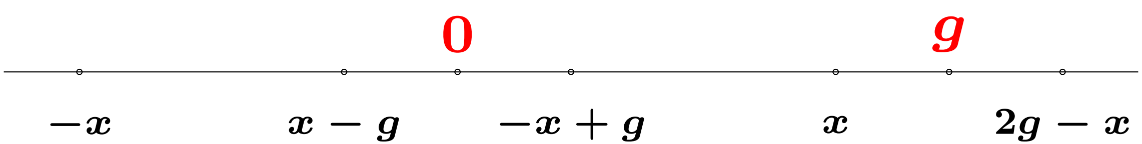

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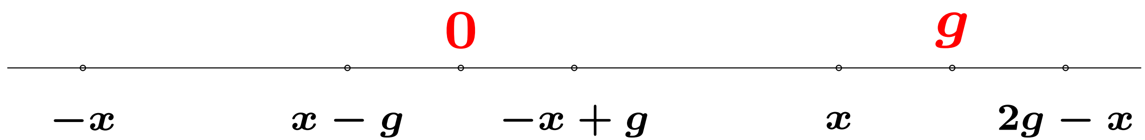

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O. Loos, *Symmetric Spaces*, 1969.

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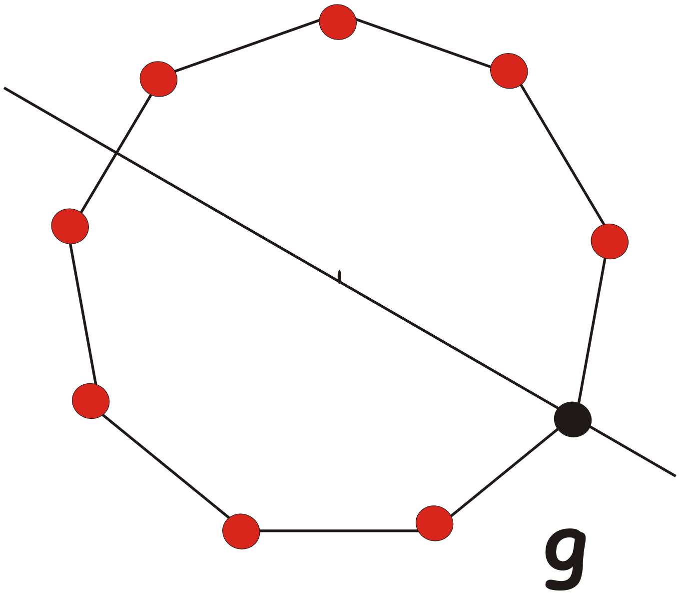
What is the number of orbits (symmetric r-ary necklaces) $s_r(G)$ and what is the number of all symmetric colorings $S_r(G)$?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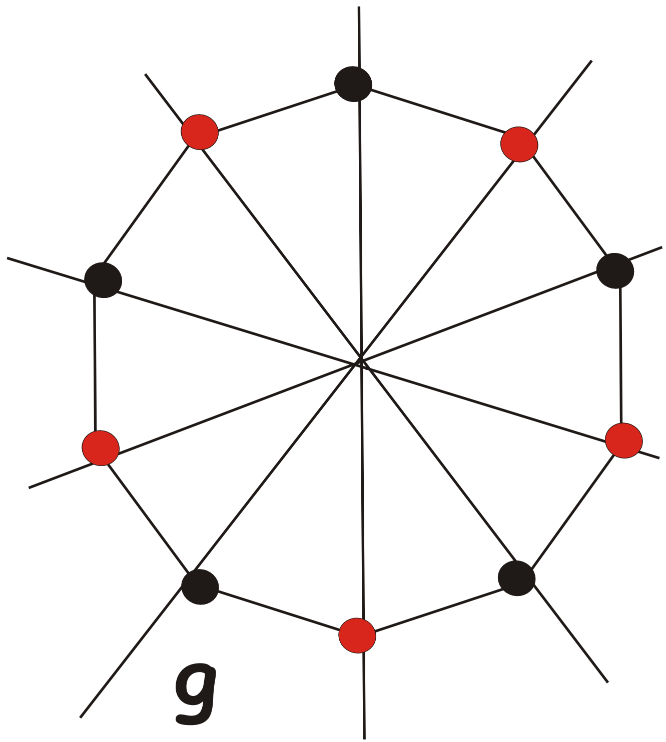
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Y. Gryshko (Zelenyuk), I. Protasov ***Symmetric colorings of finite Abelian groups*** **Dopov. Akad. Nauk Ukr., No.1 (2000), 32-33.**

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Y. Gryshko (Zelenyuk) *Symmetric colorings of regular polygons* Ars Combinatoria, 78 (2006), 277-281.

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Theorem. For every finite Abelian group A, and $r \in \mathbb{N}$.

$$B_r^*(A) = N_r^*(A) =$$

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Ye. Zelenyuk and Yu. Zelenyuk *Counting symmetric bracelets* Bull. Aust. Math. Soc., 89 (2014), 431-436.

Corollary. For all $n, r \in \mathbb{N}$, the number of symmetric r-ary necklaces is equal to the number of symmetric r-ary bracelets and is equal to

$$B_r^*(n) = N_r^*(n) = \begin{cases} \frac{1}{2}(r+1)r^{\frac{n}{2}} & \text{if } n \text{ is even} \\ r^{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

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YU. ZELENYUK, *Symmetric colorings of finite groups*, LMS LECTURE NOTE SERIES **388** (2011), 580–590.

YU. ZELENYUK ***The number of symmetric colorings of the quaternion group*** **SYMMETRY, 2 (2010), 69-75.**

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I. KASHUBA AND YU. ZELENYUK ***The number of symmetric colorings of the dihedral group D_3*** **QUAEST. MATH., 39 (2016), 65-71.**

Yu. Zelenyuk *The number of symmetric colorings of the quaternion group* **Symmetry, 2 (2010), 69-75.**

I. Kashuba and Yu. Zelenyuk *The number of symmetric colorings of the dihedral group* D_3 **Quaest. Math., 39 (2016), 65-71.**

I. Kashuba and Yu. Zelenyuk *The number of symmetric colorings of the dihedral group* D_4 **Proceedings of 4th Scientific Colloquium, Prague, (2014), 211-218.**

J. Phakathi, D. Radnell and Yu. Zelenyuk *The number of symmetric colorings of the dihedral group D_p* **Appl. Math. Inf. Sci., 10 (2016), 2373-2376.**

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J. Phakathi, W. Toko, Ye. Zelenyuk, and Yu. Zelenyuk *Symmetric colorings of the dihedral group* **Communications in Algebra, 46 (2018), 1554-1559.**

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J. Phakathi, Ye. Zelenyuk, and Yu. Zelenyuk, *Symmetric colorings of of $G \times \mathbb{Z}_2$*, submitted.

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