Genetic algorithms in constructions of block designs and SRGs

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Introduction and preliminaries

We propose a method of constructing **block designs** and **strongly regular graphs** which combines **genetic algorithm** and a method for constructing designs and strongly regular graphs with prescribed automorphism group using **orbit matrices**.

We apply this method to construct new Steiner systems with parameters $S(2, 5, 45)$, i.e. 2-$(45, 5, 1)$ designs, new symmetric designs with parameters $(71, 15, 3)$ and new strongly regular graphs with parameters $(96, 19, 2, 4)$ and $(96, 20, 4, 4)$.

Introduction and preliminaries

A **block design** $\mathcal{D}$ with parameters t-(v, k, λ) is a finite incidence structure $(\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = v$ and $1 < k < v - 1$,
2. every element (block) of $\mathcal{B}$ is incident with exactly k elements (points) of $\mathcal{P}$,
3. every t distinct points in $\mathcal{P}$ are together incident with exactly λ blocks of $\mathcal{B}$.

A **Steiner system** $S(t, k, v)$ is a block design with parameters t-$(v, k, 1)$.
A block design is **symmetric** if it has the same number of points and blocks.
In a 2-(v, k, λ) design every point is incident with exactly $r = \frac{\lambda(v-1)}{k-1}$ blocks, and r is called the replication number of a design. The number of blocks in a block design is denoted by b.

Orbit matrices of 2-designs

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a 2-(v, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$. Further, let the group G act on $\mathcal{D}$ with m point orbits and n block orbits, denoted by $\mathcal{P}_1, \ldots, \mathcal{P}_m$ and $\mathcal{B}_1, \ldots, \mathcal{B}_n$, respectively. We put $|\mathcal{P}_i| = \nu_i$ and $|\mathcal{B}_j| = \beta_j,\ i = 1, \ldots, m,\ j = 1, \ldots, n$. We denote by a_{ij} the number of blocks of $\mathcal{B}_j$ which are incident with a representative of the point orbit $\mathcal{P}_i$. The number a_{ij} does not depend on the choice of a point $P \in \mathcal{P}_i$. A decomposition of the point set and the block set with this property is called **tactical**. The following equalities hold:

(1) $0 \leq a_{ij} \leq \beta_j, \quad 1 \leq i \leq m,\ 1 \leq j \leq n,$

(2) $\sum_{j=1}^{n} a_{ij} = r, \quad 1 \leq i \leq m,$

(3) $\sum_{i=1}^{m} \frac{\nu_i}{\beta_j} a_{ij} = k, \quad 1 \leq j \leq n,$

(4) $\sum_{j=1}^{n} \frac{\nu_t}{\beta_j} a_{sj}\ a_{tj} = \lambda \nu_t + \delta_{st}(r - \lambda), \quad 1 \leq s, t \leq m,$

where $\sum_{i=1}^{m} \nu_i = v,\ \sum_{j=1}^{n} \beta_j = b$ and $b = \frac{vr}{k}$.

Orbit matrices of 2-designs

Definition

An $(m \times n)$-matrix (a_{ij}) with entries satisfying conditions $(1) - (4)$ is called a (point) **orbit matrix** for the parameters (v, k, λ) and orbit lengths distributions $(\nu_1, \dots, \nu_m)$ and $(\beta_1, \dots, \beta_n)$.

Orbit matrices are often used in the construction of designs with a presumed automorphism group. The construction of designs admitting an action of a presumed automorphism group consists of the following two basic steps:

1. Construction of orbit matrices for the given automorphism group;
2. Construction of block designs for the orbit matrices obtained in this way. This step is often called an indexing of orbit matrices.

Orbit matrices of 2-designs

The goal of the second step of the construction (indexing) is to construct incidence matrices of block designs that correspond to the orbit matrices obtained in the first step.
The indexing of orbit matrices is usually performed by exhaustive search. However, sometimes the exhaustive search is not feasible because there are too many possibilities to check.
In order to overcome this obstacle, we propose using a genetic algorithm in this step of the construction.

Example

- The incidence matrix of a symmetric 2-$(7,3,1)$ design:

$$\left[\begin{array}{c|ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

- Corresponding orbit matrix induced by Z_3 group action:

	1	3	3
1	0	3	0
3	1	1	1
3	0	1	2

Genetic algorithm

Genetic algorithms (GA) are search and optimization heuristic population based methods which are inspired by the natural evolution process. In each step of the algorithm, a subset of the whole solution space, called **population**, is being treated. The population consists of **individuals**, and each individual has **genes** that can be mutated and altered.
Every individual represents a possible solution (optimum), which is evaluated using the **fitness function**. In each iteration of the algorithm, a certain number of best-ranked individuals - **parents** is selected to create new better individuals - **children**. Children are created by a certain type of recombination - **crossover** and they replace the worst-ranked individuals in the population, providing convergence to the local optimum. After children are obtained, a **mutation** operator is allowed to occur and the next generation of the population is created. The process is iterated until the evolution condition terminates.

Combining OM and GA to construct block designs

The **individuals** in our population are incidence matrices of 1-designs admitting the action of the group G with respect to the orbit matrix M. So, individuals are $(0,1)$-matrices of type $v \times b$, the sum of entries of each row is r, the sum of entries of each column is k, admitting the action of the group G with the orbit lengths distributions $(\nu_1, \dots, \nu_m)$ and $(\beta_1, \dots, \beta_n)$ that produce the orbit matrix M.
Our aim is to take an initial population of such individuals and use a genetic algorithm to produce an individual that is the **incidence matrix** of a 2-(v, k, λ) design.

We define the **fitness function**. For every two distinct points P_i and P_j, we set a_{ij} to be the number of appearances of P_i and P_j in a common block, i.e., the dot product of the corresponding rows of the incidence matrix. Our fitness function is

$$\sum_{P_i, P_j \in \mathcal{P}} min\{a_{ij}, \lambda\},$$

where $\mathcal{P}$ is the set of points of a design.
An individual is an incidence matrix of a 2-(v, k, λ) design if and only if the value of the fitness function is $\binom{v}{2}\lambda$.

Combining OM and GA to construct block designs

Genes of an individual are the rows of the incidence matrix corresponding to point orbits.
The **crossover** is defined in a way that the genes at some positions of the first parent are replaced with the genes at the same positions of the second parent, and vice versa.
The **mutation** is performed in a way that one or more bits in a gene of an individual are replaced with other bits.
Sometimes a population gets stuck in a **local optimum**, causing a stagnation. In order to escape from a local optimum, we **reset** the algorithm. In our algorithm we have two kinds of resets, complete and partial, depending on the behaviour of the population.

Example: 2-(11,5,2) design with Z_5 group action

	1	5	5
1	0	5	0
5	1	2	2
5	0	2	3

orbit matrix

0	1	1	1	1	1	0	0	0	0	0
1	0	0	1	1	0	1	1	0	0	0
1	0	0	0	1	1	0	1	1	0	0
1	1	0	0	0	1	0	0	1	1	0
1	1	1	0	0	0	0	0	0	1	1
1	0	1	1	0	0	1	0	0	0	1
0	0	1	0	0	1	0	1	1	0	1
0	1	0	1	0	0	1	0	1	1	0
0	0	1	0	1	0	0	1	0	1	1
0	0	0	1	0	1	1	0	1	0	1
0	1	0	0	1	0	1	1	0	1	0

individual

Construction of new Steiner systems $S(2,5,45)$

Up to our best knowledge, 30 Steiner systems $S(2,5,45)$ are known. In Table 1 we give information about the full automorphism groups and 2-ranks of these previously known Steiner systems $S(2,5,45)$.

$\lvert Aut(\mathcal{D})\rvert$	$Aut(\mathcal{D})$ structure	2-rank	frequency
360	$(Z_{15} \times Z_3) : Q_8$	45	1
160	$(E_{16} : Z_5) : Z_2$	36	1
72	$E_9 : Q_8$	45	3
72	$E_9 : Q_8$	37	3
40	$Z_5 : Q_8$	45	1
32	$E_{16} : Z_2$	36	1
24	$Z_2 \times A_4$	38	3
8	Q_8	45	1
8	Q_8	37	2
6	S_3	37	2
4	Z_4	37	6
2	Z_2	37	2
1	I	37	4

Table: Previously known Steiner systems $S(2,5,45)$

Construction of new Steiner systems $S(2,5,45)$

Using the algorithm described, we have managed to construct 35 new Steiner systems $S(2,5,45)$ from orbit matrices for the action of group Z_2. Orbit matrices that we used for this process are orbit matrices created from 26 known Steiner systems $S(2,5,45)$, and then also from these new ones we constructed. Information about these new designs are presented in Table 2.

$\lvert \mathrm{Aut}(\mathcal{D}) \rvert$	$\mathrm{Aut}(\mathcal{D})$ structure	2-rank	frequency
32	$E_{16} : Z_2$	36	2
16	$(Z_4 \times Z_2) : Z_2$	39	4
16	$(Z_4 \times Z_2) : Z_2$	38	8
16	$Z_2 \times D_8$	38	2
8	E_8	37	3
8	E_8	38	7
8	E_8	39	1
8	$Z_4 \times Z_2$	39	4
4	E_4	39	4

Table: New Steiner systems $S(2,5,45)$

Construction of new 2-$(71, 15, 3)$ designs

Up to our best knowledge, 146 2-$(71, 15, 3)$ designs i.e. triplanes of order twelve admitting an action of an automorphism of order six and 2 triplanes of order twelve that do not admit an action of an automorphism of order three were known up to now. Information on these 148 designs is given in the table below.

$\|Aut(\mathcal{D})\|$	$Aut(\mathcal{D})$ structure	frequency
336	$(E_8 : F_{21}) \times Z_2$	6
168	$E_8 : F_{21}$	3
48	$E_4 \times A_4$	26
42	$F_{21} \times Z_2$	6
24	$A_4 \times Z_2$	89
24	$S_3 \times E_4$	16
8	E_8	2

Table: Known triplanes of order twelve

Construction of new 2-$(71, 15, 3)$ designs

Using the algorithm described, we have managed to construct 22 new 2-$(71, 15, 3)$ designs i.e. triplanes of order twelve from orbit matrices for the action of group Z_2. Orbit matrices that we used for this process are orbit matrices created from 148 known triplanes of order twelve, and then also from these new ones we constructed. Information about these new designs are presented in the table below.

$\|\mathrm{Aut}(\mathcal{D})\|$	$\mathrm{Aut}(\mathcal{D})$ structure	frequency
16	E_{16}	14
8	E_8	8

Table: New triplanes of order twelve

Among these designs, two of them with full automorphism groups of order 8 are self-dual.

SRGs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a simple k-regular graph with v vertices. We say that $\mathcal{G}$ is a **strongly regular graph** with parameters (v, k, λ, μ) if the number of common neighbors of every two distinct vertices x and y is

- λ if they are adjacent,
- μ if they are non-adjacent.

Strongly regular graphs with parameters (v, k, λ, μ) are denoted by $SRG(v, k, \lambda, \mu)$.

$SRG(96, 19, 2, 4)$ and $SRG(96, 20, 4, 4)$

Known $SRG(96, 19, 2, 4)$ and $SRG(96, 20, 4, 4)$ were constructed by Haemers, Wallis, Muzychuk[1], Brouwer, Koolen and Klin[2], and Braić, Golemac, Mandić and Vučičić[3]. Sizes of full automorphism groups of these known $SRG(96, 19, 2, 4)$ range from 96 to 9 216 of $SRG(96, 20, 4, 4)$ range from 16 to 138 240.
Ihringer[4] constructed more SRGs with these parameters which have full automorphism groups of smaller sizes (including trivial ones).

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[1] M. Muzychuk, A generalization of Wallis-Fon-Der-Flaass construction of strongly regular graphs, J. Algebr. Comb. 25 (2007) 169-187.

[2] A. E. Brouwer, J. H. Koolen and M. H. Klin, A root graph that is locally the line graph of the Petersen graph, Discr. Math. 264 (2003) 13-24.

[3] S. Braić, A. Golemac, J. Mandić, T. Vučičić, Graphs and symmetric designs corresponding to difference sets in groups of order 96, Glasnik Matematički Vol. 45(65)(2010), 1-14.

[4] http://math.ihringer.org/srgs.php

New $SRG(96, 19, 2, 4)$ and $SRG(96, 20, 4, 4)$

We constructed 21 $SRG(96, 19, 2, 4)$ out of which 15 are new and 22 new $SRG(96, 20, 4, 4)$.

$\|Aut(\mathcal{G})\|$	$Aut(\mathcal{G})$ structure	frequency
8	E_8	2
4	Z_4	1
4	E_4	6
2	Z_2	12

Table: $SRG(96, 19, 2, 4)$

$\|Aut(\mathcal{G})\|$	$Aut(\mathcal{G})$ structure	frequency
16	$Z_2 \times D_8$	1
8	E_8	3
4	E_4	8
2	Z_2	10

Table: $SRG(96, 20, 4, 4)$

Thank you!